Isometric embedding of Busemann surfaces into L_1

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Abstract. In this note, we prove that any non-positively curved 2-dimensional surface (alias, Busemann surface) is isometrically embeddable into L_1. As a corollary, we obtain that all planar graphs which are 1-skeletons of planar non-positively curved complexes with regular Euclidean polygons as cells are L_1-embeddable with distortion at most $2 + \pi/2 < 4$. Our results significantly improve and simplify the results of the recent paper *A. Sidiropoulos, Non-positive curvature, and the planar embedding conjecture, FOCS 2013.*

1. Avant-propos

Isometric and low distortion embeddings of finite and infinite metric spaces into L_p-spaces is one of the main subjects in the theory of metric spaces. Work in this area was initiated by Cayley in 1841 and continued in the first half of the 20th century by Fréchet, Menger, Schoenberg, and Blumenthal. Since these days it is known that all metric spaces isometrically embed into L_∞. Metric spaces isometrically embeddable into L_2 were characterized by Menger and Schoenberg. Even if embeddability into L_1 can be defined in several equivalent ways and a few necessary conditions for L_1-embedding are known, metric spaces isometrically embeddable into L_1 cannot be characterized in an efficient way, because deciding whether a finite metric space is L_1-embeddable is NP-complete. On the other hand, many classes of metric spaces (Euclidean and spherical metrics, tree and outerplanar metrics, as well as graph metrics of some classes of graphs) are known to be L_1-embeddable; for a full account of the theory of the isometric embeddings, see the book [9].

Although already simple metric spaces are not L_1-embeddable, Bourgain [4] established that any metric space on n points can be embedded into L_1 with $O(\log n)$ (multiplicative) distortion and this important result has found numerous algorithmic applications (for a theory of low distortion embeddings of metric spaces and its algorithmic applications, the interested reader can consult the book [15] and the survey [13]). One of main open problems in this domain is the so-called *planar embedding conjecture* asserting that all planar metrics (i.e., metrics of planar graphs) can be embedded into L_1 with constant distortion. This conjecture was established for series-parallel graphs [12]; on the other hand, several classes of planar graphs are known to be L_1-embeddable (see the book [9] and the survey [2]), in particular, it was shown in [7] that the three basic classes of non-positively curved planar graphs (so-called, (3,6),(6,3), and (4,4)-graphs) are L_1-embeddable.

Recently, Sidiropoulos [17] proved that for any finite set Q of a non-positively curved planar surface (S,d), the metric space (Q,d) is L_1-embeddable with constant distortion. As a consequence, all planar graphs which give rise to non-positively curved surfaces can be embedded into L_1 with constant distortion.[1] The proof-method in [17] uses at minimum the geometry of (S,d) and essentially employs probabilistic techniques. Using the convexity of the distance function, Sidiropoulos "approximates" (Q,d) by special planar graphs called "bundles", which he shows to be L_1-embeddable with constant distortion. To provide such an embedding, the author searches for a good distribution over spacial types of cuts (bipartitions of Q), called "monotone cuts", defined on bundles. Searching for this good distribution is the most technically involved part of the paper [17].

In this note, we prove that in fact all non-positively curved planar surfaces (S,d) (which we call *Busemann surfaces*) are L_1-embeddable (without any distortion). This significantly improves the result of [17]. Our approach is geometric and combinatorial. First, we establish some elementary properties of convexity in Busemann surfaces, analogous to the properties of usual convexity in $\mathbb{R}^2$. Then we use some of these properties to show that for any finite set Q in general position of S, (Q,d) is L_1-embeddable. For this, we extend to Busemann surfaces the proof-method of a combinatorial Crofton lemma given by R. Alexander [1] for finite point-sets in general position in $\mathbb{R}^2$ endowed with a metric in which lines are geodesics. Using local perturbations of points, we extend our result to all finite sets Q of S. Then the fact that (S,d) is L_1-embeddable follows from a compactness result of [5] about L_p-embeddings.

2. Preliminaries

2.1. **L_1-embeddings.** A metric space (X,d) is *isometrically embeddable* into a metric space (X',d') if there exists a map $\varphi : X \to X'$ such that $d'(\varphi(x),\varphi(y)) = d(x,y)$ for any $x,y \in X$. More generally, $\varphi : X \mapsto X'$ is an *embedding with (multiplicative) distortion* $c \geq 1$ if $d(x,y) \leq d'(\varphi(x),\varphi(y)) \leq c \cdot d(x,y)$ for all $x,y \in X$ (non-contractive embedding), or if $\frac{1}{c} \cdot d(x,y) \leq d'(\varphi(x),\varphi(y)) \leq d(x,y)$ for all $x,y \in X$ (non-expansive embedding). Let $(\Omega,\mathcal{A},\mu)$ be a measure space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of subsets of Ω, and a measure μ on $\mathcal{A}$. Given a function $f : \Omega \to \mathbb{R}$, its L_1-*norm* is defined by $\|f\|_1 = \int_\Omega |f(w)|\mu(dw)$. Then $L_1(\Omega,\mathcal{A},\mu)$ denotes the set of functions $f : \Omega \to \mathbb{R}$ which satisfy $\|f\|_1 < \infty$. The L_1-norm defines a metric on $L_1(\Omega,\mathcal{A},\mu)$ by taking $\|f-g\|_1$ as the distance between two functions $f,g \in L_1(\Omega,\mathcal{A},\mu)$. A metric space (X,d) is said to be L_1–*embeddable* if there exists an isometric embedding of (X,d) into $L_1(\Omega,\mathcal{A},\mu)$ for some measure space $(\Omega,\mathcal{A},\mu)$ [9]. If Ω is finite (say, $|\Omega| = n$) and $\mathcal{A} = 2^\Omega$, the resulting space $L_1(\Omega,\mathcal{A},\mu)$ is the n-dimensional l_1-space $(\mathbb{R}^n,d_1)$, where the l_1-distance between two points $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ is $d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$. A metric space (X_n,d) on n points is l_1-*embeddable* (and d is called an l_1-*metric*) if there exists an isometric embedding of (X_n,d) into some l_1-space $(\mathbb{R}^m,d_1)$. It is well known [9, Chapter 4] that the set of all l_1-metrics on X_n form a closed cone CUT_n in $\mathbb{R}^{\frac{n(n-1)}{2}}$, called the *cut cone*. CUT_n is generated by the cut semimetrics δ_S for $S \subseteq X_n$, where $\delta_S(x,y) = 1$ if $|S \cap \{x,y\}| = 1$ and $\delta_S(x,y) = 0$ otherwise. A well–known compactness result

[1] Non-positively curved metric spaces constitute a large class of geodesic metric spaces at the heart of modern metric geometry and playing an essential role in geometric group theory [6, 16].

of [5] implies that L_1–embeddability of a metric space is equivalent to l_1–embeddability of its finite subspaces.

2.2. **Geodesics and geodesic metric spaces.** In this subsection, we recall some definitions and notations on geodesic metric spaces; we closely follow the books [6] and [16]. Let (X, d) be a metric space. A *path* in X is a continuous map $\gamma : [a, b] \to X$, where a and b are two real numbers with $a \leq b$. If $\gamma(a) = x$ and $\gamma(b) = y$, then x and y are the *endpoints* of γ and that γ joins x and y. A *geodesic path* (or simply a *geodesic*) in X is a path $\gamma : [a, b] \to X$ that is distance-preserving, that is, such that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. A *geodesic line* (or simply a *line*) is a distance-preserving map $\gamma : \mathbb{R} \to X$ and a *geodesic ray* (or simply a *ray*) is a distance-preserving map $\gamma : [0, \infty[\to X$. A path $\gamma : [a, b] \to X$ is said to be a *local geodesic* if for all t in (a, b) one can find a closed interval $I(t) \subseteq [a, b]$ containing t in its interior such that the restriction of γ on $I(t)$ is geodesic. A metric space X is *geodesic* if every pair of points in X can be joined by a geodesic. A *uniquely geodesic space* is a geodesic space in which every pair of points can be joined by a unique geodesic.

2.3. **Non-positively curved spaces.** We continue with the definitions of non-positively curved spaces in the sense of Alexandrov [6] and of Busemann [16]. A *geodesic triangle* $\Delta := \Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic between each pair of vertices (the edges of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is a *CAT(0) space* (or a *non-positively curved space in the sense of Alexandrov*) [11] if for all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X, if y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2$, then $d(x_3, y) \leq d_{\mathbb{E}^2}(x'_3, y')$. CAT(0) spaces have many fundamental properties and can be characterized in several natural ways (for a full account of this theory consult the book [6]).

A *Busemann space* (or a *non-positively curved space in the sense of Busemann*) is a geodesic metric space (X, d) in which the distance function between any two geodesics is convex: for any two reparametrized geodesics $\gamma : [a, b] \to X$ and $\gamma' : [a', b'] \to X$ the map $f_{\gamma,\gamma'}(t) : [0, 1] \to \mathbb{R}$ defined by $f_{\gamma,\gamma'}(t) = d(\gamma((1 - t)a + tb), \gamma'((1 - t)a' + tb'))$ is convex. Each CAT(0) space is a Busemann space, but non vice-versa. However, Busemann spaces still satisfy most of fundamental properties of CAT(0) spaces: they are contractible, uniquely geodesic, local geodesics are geodesics, and geodesics vary continuously with their endpoints (for these and other results on Busemann spaces consult the book [16]). Busemann spaces and CAT(0) spaces are the same in the case of smooth Riemannian manifolds and of piecewise Euclidean or hyperbolic complexes (because in the latter case, the CAT(0) property is equivalent to the uniqueness of geodesics; see [6, Theorem 5.4]).

2.4. **Busemann surfaces.** A *planar surface* S is a 2-dimensional manifold without boundary, i.e., S is homeomorphic to the plane $\mathbb{R}^2$. A geodesic metric space (S, d) is called a *Busemann surface* if S is a planar surface and the metric space (S, d) is a Busemann space.

Notice that each point x of S has an open neighborhood $B(x, \epsilon)$ which is homeomorphic to an open ball in the plane.

Particular instances of Busemann surfaces are non-positively curved piecewise-Euclidean (PE) (or piecewise hyperbolic) planar complexes without boundary. In fact, as we will show below, any finite non-positively curved planar complex can be extended to a Busemann surface. Recall that a *planar PE complex* X is obtained from a (not necessarily finite) planar graph G by replacing each inner face of G having n sides by a convex n-gon in the Euclidean plane. Then X is called a *regular planar complex* if each face of G with n sides is replaced by a regular n-gon in the plane. Note that the graph G is the 1-skeleton of X. The complex X is called a *non-positively curved planar complex* if the sum of angles around each inner vertex of G is at least 2π or, equivalently, if X endowed with the intrinsic l_2-metric is uniquely geodesic. We will call a planar graph G a *Busemann graph* (or a *non-positively curved planar graph*) if G is the 1-skeleton of a regular non-positively curved planar complex. Basic examples of Busemann graphs are so-called (3,6),(4,4), and (6,3)-graphs (a planar graph G embedded into the plane is called a *(p,q)-graph* if the degrees of all inner vertices are at least p and all inner faces have lengths at least q). It was shown in [7] (see also [2, Proposition 8.6]) that all (3,6),(4,4), and (6,3)-graphs are l_1-embeddable (for other properties of these graphs, see [3]). We will present below a Busemann planar graph which is not l_1-embeddable.

To embed a finite non-positively curved planar complex X into a Busemann surface S, to each boundary edge e of X we add a closed halfplane H_e of $\mathbb{R}^2$ so that e is a segment of the boundary of H_e. If two boundary edges e, e' of X share a common endvertex x, then H_e and H'_e will be glued along the rays of their boundaries emanating from x which are disjoint from e and e'. It can be easily seen that the resulting planar surface S is CAT(0) and that X isometrically embeds into S.

2.5. **Main results.** We continue with the formulation of main results of this note:

Theorem 1. *If (S, d) is a Busemann surface, then (S, d) is L_1-embeddable.*

Corollary 1. *Any Busemann graph G endowed with its standard-graph metric d_G a non-expansive L_1-embedding with distortion at most $2 + \pi/2$.*

To prove Theorem 1, by a compactness result of [5], it suffices to show that for any finite subset Q of S, (Q, d) is l_1-embeddable. For this, first we show that any finite subset of S in general position (no three points on a common geodesic line) is l_1-embeddable. Then, using this result and a local perturbations of points, we prove that there exists a sequence d_i of l_1-metrics on Q converging to d. Since the cone of l_1-metrics on Q is closed [9], (Q, d) is l_1-embeddable as well. Our proof that any finite set of S in general position is l_1-embeddable is based on a beautiful Crofton formula by Alexander [1] established for finite point-sets in general position of $\mathbb{R}^2$ endowed with a metric in which lines are geodesics (for another use of this formula, see [8]). To generalize this result to Busemann surfaces S, we extend to S some elementary properties of the usual convexity in the plane (see also [14] for some similar properties of planar CAT(0) complexes and [10] for such properties for topological affine planes). This is done in Section 3. The proof of the Crofton formula for Busemann surfaces is given in Section 4 and the proof of the main results is completed in Section 5.

3. Geodesic lines and convexity

In this section, we present elementary properties of geodesic lines and convex sets in Busemann planar surfaces (S, d). For two points $x, y \in S$, we denote by $[x, y]$ the unique geodesic segment joining x and y. A set of points Q of S is in *general position* if no three points of Q are collinear, i.e., lie on a common line of S. For three points x, y, z of S, we will denote by $\Delta^*(x, y, z)$ the closed region of S bounded by the geodesics $[x, y], [y, z], [z, x]$ of the geodesic triangle $\Delta(x, y, z)$. A set $R \subseteq S$ is called *convex* if $[p, q] \subseteq R$ for any $p, q \in R$. For a set Q of S we denote by $\text{conv}(Q)$ the smallest convex set containing Q and call $\text{conv}(Q)$ the *convex hull* of Q.

3.1. **Geodesic lines.** A geodesic metric space (X, d) is said to have the *geodesic extension property* if for every local geodesic $\gamma : [a, b] \to X$, with $a < b$, there exists $\epsilon > 0$ and a local geodesic $\gamma' : [a, b + \epsilon] \to X$ such that $\gamma'|_{[a,b]} = \gamma$. Since in Busemann spaces local geodesics are geodesics, for such spaces the geodesic extension property is equivalent to the fact that the geodesic between any two distinct points can be extended to a geodesic line. It was established in Proposition 5.12 of [6] that any CAT(0) space that is homeomorphic to a finite dimensional manifold has the geodesic extension property. By [6, Footnote 24] and the proof of this proposition, an analogous statement holds for Busemann spaces. Therefore, for Busemann surfaces (S, d) we obtain:

Lemma 1. *S has the geodesic extension property.*

The following lemma immediately follows from the definition of Busemann spaces.

Lemma 2. *Closed balls of S are convex.*

Lemma 3. *Any geodesic line ℓ partitions S into two connected components.*

Proof. Let x be a point of ℓ and let $B(x, \epsilon)$ be a closed ball centered at x. Since $B(x, \epsilon)$ is convex and S is a planar surface, $B(x, \epsilon) \cap \ell = [p, q]$ and the segment $[p, q]$ partitions $B(x, \epsilon)$ in two connected components. Let u, v be two points from different components of $B(x, \epsilon/2) \setminus [p, q]$. By Lemma 2, $p \notin [u, v]$. Since $B(x, \epsilon)$ is convex, necessarily $[u, v]$ intersects $[p, q]$; let $x' \in [u, v] \cap [p, q]$. Now, if ℓ does not separate S, then u and v can be connected by a path γ not intersecting ℓ. Let R be the region of S bounded by γ and $[u, v]$. Then one of the points p, q, say p, belongs to R. The ray $r_{x'}$ of ℓ emanating from x' and passing via p enters the region R, hence $r_{x'}$ must intersect the boundary of R in a point w different from x'. Then $p \in [x', w]$. Since $\ell \cap \gamma = \varnothing$, necessarily $w \in [u, v]$ and we conclude that x' and w are joined by two different geodesics, one is a portion of ℓ passing via $p \notin [u, v]$ and the second is a portion of $[u, v]$, a contradiction. $\square$

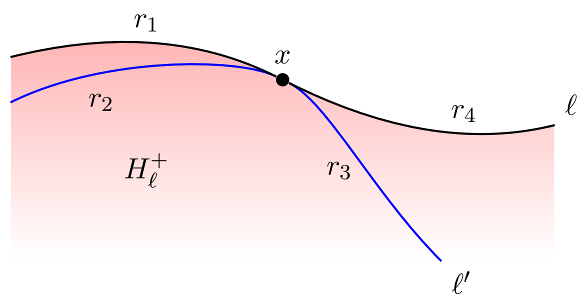

FIGURE 1. Illustration to Lemma 5.

For a line ℓ, we denote by $\mathring{H}_\ell^-$ and $\mathring{H}_\ell^+$ the two connected components of $S \setminus \ell$, respectively, and call them *open halfplanes*. The *closed halfplanes* defined by ℓ are the sets $H_\ell^- = \mathring{H}_\ell^- \cup \ell$ and $H_\ell^+ = \mathring{H}_\ell^+ \cup \ell$. Since each line ℓ is convex, the following result is straightforward.

Lemma 4. *The closed halfplanes H_ℓ^- and H_ℓ^+ are convex sets of S.*

Lemma 5. *Let ℓ and ℓ' be two intersecting geodesic lines such that ℓ' is contained in the closed halfplane H_ℓ^+ defined by ℓ, $x \in \ell \cap \ell'$, and let $r_1, \dots r_4$ be the four rays emanating from x defined as in Figure 1. Then $r_1 \cup r_3$ and $r_2 \cup r_4$ are also geodesic lines.*

Proof. We will prove that $\ell_0 = r_2 \cup r_4$ is a geodesic line. If $\ell \cap \ell' = [x', x'']$ with $x' \neq x''$, then ℓ_0 is a local geodesic because it is covered by the rays of ℓ and ℓ' emanating from x' and x'' sharing the geodesic segment $[x', x'']$. Thus ℓ_0 is a geodesic line.

Now, suppose that $\ell \cap \ell' = \{x\}$. Pick any two points $y \in r_2$ and $z \in r_4$ and suppose by way of contradiction that $[y, z]$ is not contained in ℓ_0. Moreover, we can suppose without loss of generality that $\ell_0 \cap [y, z] = \{y, z\}$. But then $[y, z]$ necessarily intersects one of the rays r_1 or r_3, say there exists $z' \in [y, z] \cap r_3$. Then we obtain two distinct geodesics between y and z': one along ℓ' and the second along $[y, z]$. $\square$

Lemma 6. *(Pasch axiom) If $\Delta(x, y, z)$ is a geodesic triangle, $u \in [x, y]$, and $z \in [x, v]$, then $[u, v] \cap [y, z] \neq \varnothing$ (see Figure 2(a)).*

Proof. The assertion is obvious if $u \in \{x, y\}$ or $v \in [y, z]$. So, let $u \notin \{x, y\}$ and $v \notin [y, z]$. If there exists a point $v' \in [u, v] \cap [x, z]$, then $z \in [u, v]$ by convexity of $[u, v] \cap [x, v]$. So, we can further suppose that $[u, v] \cap [z, x] = \varnothing$.

Note that $v \notin \Delta^*(x, y, z)$. Indeed, if $v \in \Delta^*(x, y, z)$, consider any line ℓ extending $[x, v]$. The ray from ℓ emanating from v and not containing x intersects $[x, y]$ or $[y, z]$. But in this case $\ell \cap [x, y]$ or $\ell \cap [y, z]$ is not convex.

Let ℓ be a line extending $[x, y]$. Let r_x and r_y be the two disjoint rays of ℓ emanating from x and y, respectively. $[u, v]$ intersects $r_y \cup [y, z] \cup [z, x] \cup r_x$. Because $[u, v] \cap l$ is convex and does not contain x nor y, $[u, v] \cap (r_x \cup r_y) = \varnothing$. Therefore $[u, v] \cap [y, z] \neq \varnothing$. $\square$

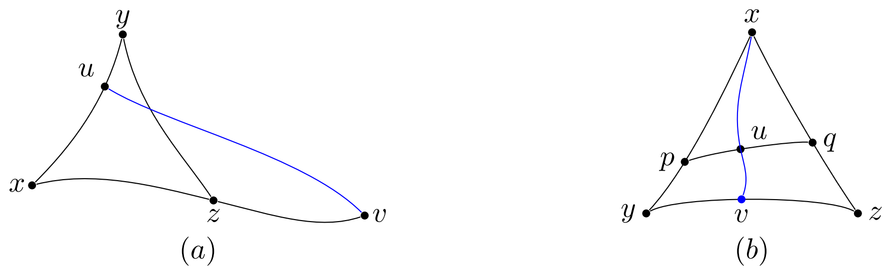

FIGURE 2. Pasch and Peano axioms.

For two distinct points $x, y \in S$, let $C(x,y) = \{z \in S : x \in [y,z]\}$ and $C(y,x) = \{z \in S : y \in [x,z]\}$; we will call the sets $C(x,y)$ and $C(y,x)$ *cones*. Since S satisfies the geodesic extension property, the set $C(x,y) \cup [x,y] \cup C(y,x)$ can be equivalently defined as the union of all geodesic lines extending $[x,y]$.

Lemma 7. *$C(x,y)$ and $C(y,x)$ are convex and closed subsets of S.*

Proof. Let $u, v \in C(x,y)$ and $w \in [u,v], w \notin \{u,v\}$. By Pasch axiom, there exists a point $w' \in [y,w] \cap [u,x]$. Since $w' \in [u,x] \subset C(x,y)$, we conclude that $x \in [w',y] \subset [w,y]$, whence $w \in C(x,y)$. This shows that $C(x,y)$ is convex. To show that $C(x,y)$ is closed, let $\{u_i\}$ be a sequence of points of $C(x,y)$ converging to a point $u \in S$. Since $\{d(y,u_i)\}$ converges to $d(y,u)$, $\{d(x,u_i)\}$ converges to $d(x,u)$, and $d(y,x) + d(x,u_i) = d(y,u_i)$, we conclude that $d(y,x) + d(x,u) = d(y,u)$, hence $u \in C(x,y)$. □

Lemma 8. *$\Delta^* := \Delta^*(x,y,z)$ is the convex hull of $\{x,y,z\}$.*

Proof. First we show that Δ^* is convex. Suppose by way of contradiction that for two points $p, q \in \Delta^*$, $[p,q]$ contains a point $s \in S \setminus \Delta^*$. Without loss of generality, we can assume that $[p,q] \cap \Delta^* = \{p,q\}$ and that $p \in [x,y]$, $q \in [x,z]$. Let ℓ be a line extending $[x,y]$. By convexity of closed halfplanes, $[p,q]$ and z must be in the same halfplane defined by line, thus, since $[p,q] \cap \Delta^* = \{p,q\}$, the only possible positions for p are $p \in \{x,y\}$. Similarly, $q \in \{x,z\}$. This implies that $s \in \Delta \subset \Delta^*$, a contradiction.

Now, we show that $\Delta^* \subseteq \mathrm{conv}(x,y,z)$. Notice that $\Delta(x,y,z) \subseteq \mathrm{conv}(x,y,z)$. Let $w \in \Delta^* \setminus \Delta(x,y,z)$. Let ℓ be any line containing w. The two rays of ℓ emanating from w must each intersect Δ, say in u and v respectively. Hence $w \in [u,v] \subset \mathrm{conv}(x,y,z)$. □

3.2. **Geodesic convexity.** In this subsection, we establish some elementary properties of convex sets in Busemann surfaces (S,d).

Lemma 9. *(Peano axiom) If $\Delta(x,y,z)$ is a geodesic triangle, $p \in [x,y]$, $q \in [x,z]$, and $u \in [p,q]$, then there exists a point $v \in [y,z]$ such that $u \in [x,v]$. (see Figure 2(b))*

Proof. We can suppose that $u \neq \Delta(x,y,z)$, otherwise the result is obvious. By Lemma 8, the point u belongs to $\Delta^*(x,y,z)$. Let ℓ be a geodesic line extending $[x,u]$ and let r_x be the ray of ℓ emanating from x and passing via u. Necessarily r_x will intersect one of the sides of $\Delta^*(x,y,z)$. Since $u \notin [x,y] \cup [x,z]$, $u \in r_x$, and S is uniquely geodesic, r_x necessarily intersects $[y,z]$ in a point v. Then $u \in [x,v]$, and we are done. □

It is well known [18] that Peano axiom is equivalent to the following property, called *join-hull commutativity*: for any convex set A and any point $p \notin A$, $\mathrm{conv}(p \cup A) = \cup\{[p,x] : x \in A\}$. As a consequence, we obtain:

Lemma 10. *The geodesic convexity of S is join-hull commutative.*

We continue by establishing that the Caratheodory number of geodesic convexity of S equals 3:

Lemma 11. *For any finite set Q of S,* $\mathrm{conv}(Q) = \cup\{\Delta^*(x,y,z) : x,y,z \in Q\}$.

Proof. We proceed by induction on $n = |Q|$. Let $x \in Q$ and suppose that the assertion holds for the set $Q^x = Q \setminus \{x\}$. Let $K^x = \mathrm{conv}(Q^x)$. Pick $p \in \mathrm{conv}(Q)$. If $p \in K^x$, we are done by induction assumption. So, let $p \notin K^x$. Since $\mathrm{conv}(Q) = \mathrm{conv}(x \cup K^x)$, by join-hull commutativity there exists a point $q \in K^x$ such that $p \in [x,q]$. By induction assumption, there exists three points $y,z,v \in Q^x$ such that $q \in \Delta^* = \Delta^*(y,z,v)$. The geodesic segment $[x,q]$ necessarily intersects one of the sides $[y,z],[z,v],[v,y]$ of Δ^*, say there exists $q' \in [x,q] \cap [y,z]$. But then $p \in [x,q']$ and we conclude that $p \in \mathrm{conv}(x,y,z)$ with $x,y,z \in Q$. □

Let Q be a finite set of points of S and $K := \mathrm{conv}(Q)$. Let Q_0 denote the set of all points $u \in Q$ such that u does not belong to any triangle $\Delta^*(x,y,z)$ with $x,y,z \in Q$ and $u \neq x,y,z$. By Lemma 11, $Q_0 \neq \varnothing$, moreover $\mathrm{conv}(Q_0) = \mathrm{conv}(Q)$. We call the points of Q_0 *extremal points* of Q (or of K). A line ℓ is called a *tangent line* (or, simply a *tangent*) of K if $K \cap \ell \neq \varnothing$ and K is contained in one of the closed halfplanes defined by ℓ. A geodesic segment $[x,y]$ is called an *edge* of K if $x,y \in Q$ and some line ℓ extending $[x,y]$ is a tangent of K. Clearly, each edge of K belongs to the boundary of K. A geodesic line ℓ is called a *bitangent* of two disjoint convex sets K' and K'' if ℓ is a tangent line of K' and K''. A bitangent ℓ of K',K'' is called an *inner bitangent* if K' and K'' belong to different closed halfplanes defined by ℓ.

Lemma 12. *If Q is a finite set in general position of S, then $[p,q]$ is an edge of* $K := \mathrm{conv}(Q)$ *if and only of any line ℓ extending $[p,q]$ is a tangent of K. If (Q',Q'') is a bipartition of Q such that the convex hulls* $K' = \mathrm{conv}(Q'), K'' = \mathrm{conv}(Q'')$ *are disjoint and ℓ is an inner bitangent of K',K'' such that $p' \in Q' \cap \ell$ and $p'' \in Q'' \cap \ell$, then any geodesic line extending $[p',p'']$ is an inner bitangent of K',K''.*

Proof. Let $[p,q]$ be an edge of K and let ℓ be a tangent of K extending $[p,q]$. Suppose by way of contradiction that some line ℓ' extending $[p,q]$ is not a tangent of K, i.e., two points x and y of Q belong to complementary halfplanes defined by ℓ', say y is in the region delimited by the rays of ℓ and ℓ' emanating from p that do not contain q. Then consider $[y,q]$, by convexity of the halfplanes delimited by ℓ and ℓ', $p \in [y,q]$, hence $y \in C(q,p)$, contrary to the assumption that the points of Q are in general position.

Analogously, if ℓ is an inner bitangent of K', K' as defined in the lemma, but a line ℓ' extending $[p', p'']$ is not an inner bitangent, we will conclude that some point $x \in Q$, $x \notin \{p', p''\}$ belongs to one of the cones $C(p', p'')$ or $C(p'', p')$, contrary to the assumption that the points of Q are in general position. □

Lemma 13. *For any finite set Q of S, the set Q_0 of extremal points of $K := \mathrm{conv}(Q)$ admits a circular order $\pi = (p_{i_1}, \ldots p_{i_k})$, such that the geodesic segments $[p_{i_j}, p_{i_{j+1(mod\ k)}}]$ are edges of K and the boundary of K is the union of these edges $[p_{i_j}, p_{i_{j+1(mod\ k)}}]$.*

Proof. We proceed by induction on $|Q|$. For a point $x \in Q$, let $Q^x = Q \setminus \{x\}$ and $K^x = \mathrm{conv}(K^x)$. If for some $x \in Q$, $x \in K^x$, then $x \notin Q_0$ and $\mathrm{conv}(Q) = \mathrm{conv}(Q^x)$, so we can apply the induction hypothesis to the set Q^x. Thus we can assume that all points of Q are extremal.

To define the required circular order π on Q, for each point $x \in Q$ we have to define its two neighbors in π. Consider the set Q^x. Then obviously $Q^x_0 = Q^x$ and, by the induction assumption, the points of Q^x admit the required circular order π'. We will prove now that we can find two consecutive points $y, z \in Q^x$ of π' such that $[x, y]$ and $[x, z]$ are edges of K. Then the circular order π is obtained from π' by inserting the point x between y and z.

For two distinct points $u, v \in Q^x$, let $R(u, v)$ be the closed region of S comprised between two rays of ℓ' and ℓ'' from x extending $[x, u]$ and $[x, v]$, respectively. Additionally, we choose u and v such that $n(u, v)$, the number of points of $Q \cap R(u, v)$, is maximum. Notice that for any point $w \in Q^x$, we have $w \in R(u, v)$ if and only if $[x, w] \cap [u, v] \neq \varnothing$.

Let y, z be a pair of points of Q^x for which $n(y, z)$ is maximal. We claim that $n(y, z) = |Q|$. Suppose by way of contradiction that there exists a point $y' \in Q^x$ such that some line ℓ' extending $[x, y]$ separates y' from z. We claim that $n(y', z) > n(y, z)$. Let ℓ be a geodesic line extending $[x, y']$ and defining $n(y', z)$. First, since ℓ' separates y' and z, we have $[y', z] \cap \ell' \neq \varnothing$. Since $y \notin \Delta^*(x, y', z)$, this implies that $[y', z] \cap [x, y] \neq \varnothing$, i.e., $y \in R(y', z)$. Let $u \in [y', z] \cap [x, y]$. Pick any point $w \in R(y, z)$. Then there exists $w' \in [y, z] \cap [w, x]$. Since $[u, z] \cap [w', x] \neq \varnothing$ by Pasch axiom applied to the triangle $\Delta(x, y', w)$, we conclude that $[w, x]$ intersects $[y', z]$, whence $w \in R(y', z)$. Since $y' \in R(y', z) \setminus R(y, z)$, we deduce that $n(y', z) > n(y, z)$, contrary to the maximality choice of the pair y, z. This proves that $n(y, z) = |Q|$. Therefore for the lines ℓ', ℓ'' extending $[x, y]$ and $[x, z]$, the set Q is contained in the convex set $R(y, z)$ bounded by ℓ' and ℓ''. Hence ℓ', ℓ'' are tangents of $K = \mathrm{conv}(Q)$, showing that $[x, y], [x, z]$ are edges of K.

Note that $[y, z]$ is an edge of $K^x = \mathrm{conv}(Q^x)$ (i.e., y, z are consecutive in π'), otherwise $\Delta^*(x, y, z)$ will contain yet another point of Q, contrary to the assumption that all points of Q are extremal. It remains to show that any edge $[u, v]$ of K^x different from $[y, z]$ is also an extremal edge of K. From the choice of $[y, z]$, we conclude that $[x, u] \cap [y, z] \neq \varnothing$ and $[x, v] \cap [y, z] \neq \varnothing$. Let $u' \in [x, u] \cap [y, z]$. Pick any line ℓ extending $[u, v]$; ℓ is a tangent of K^x. If ℓ is not a tangent of K, it implies that ℓ intersect $[x, y]$ and $[x, z]$. Consequently, $[x, u']$ intersects ℓ, and thus $\ell \cap [x, u]$ is not convex, a contradiction. Hence $[u, v]$ is also an edge of K. □

Lemma 14. *If (Q', Q'') is a bipartition of a finite set Q of S such that the convex hulls $K' = \mathrm{conv}(Q'), K'' = \mathrm{conv}(Q'')$ are disjoint, then there exists four (not necessarily distinct) points $p', q' \in Q'$ and $p'', q'' \in Q''$ and two inner bitangents extending $[p', p'']$ and $[q', q'']$ respectively. Any inner bitangent of K', K'' extends $[p', p'']$ or $[q', q'']$.*

Proof. For a line ℓ extending a segment $[p', p'']$ with $p' \in Q'$, $p'' \in Q''$, orient ℓ from p' to p'', and denote by H_l^+ the closed halfplane delimited by ℓ on the right of ℓ, and by H_ℓ^- the closed halfplane on the left of ℓ.

Let $p' \in Q'$ and $p'' \in Q''$ be two points such that for some line ℓ' extending $[p', p'']$ the value of $|Q' \cap H_{\ell'}^-| + |Q'' \cap H_{\ell'}^+| = n_1(p', p'')$ is as large as possible. We assert that ℓ' is an inner bitangent of K' and K''. Suppose not: then either there exists $q \in Q'' \cap \mathring{H}_{\ell'}^-$ or $p \in Q' \cap \mathring{H}_{\ell'}^+$, say the first. We will show that $n_1(p', q) > n_1(p', p'')$, leading to a contradiction with the maximality choice of the pair p', p''.

Pick any line ℓ extending $[p', q]$. Let $r_{p'}$ and r_q be the two disjoint rays of ℓ emanating from p' and q. Clearly, $r_{p'} \cap K'' = \varnothing$, otherwise by convexity of $K'' \cap \ell$, $p' \in K''$, contradiction. Analogously $r_q \cap K' = \varnothing$. Notice also that $r_q \cap \ell' = \varnothing$, otherwise by convexity of $\ell \cap \ell'$, $q \in \ell$.

First notice that $Q'' \cap H_{\ell'}^+ \subset Q'' \cap H_\ell^+$. Indeed, if this is not the case, ℓ must intersect $K'' \cap \mathring{H}_{\ell'}^+$. Since $r_{p'} \cap K'' = \varnothing$ and $r_q \in \mathring{H}_{\ell'}^-$, necessarily $[p', q]$ intersects $K'' \cap \mathring{H}_{\ell'}^+$. But this is not possible by convexity of $H_{\ell'}^-$ which contains both p' and q. This establishes the required inclusion. If $Q' \cap H_{\ell'}^- \subset Q' \cap H_\ell^-$, then since $q \in Q'' \cap H_\ell^+ \setminus Q'' \cap H_{\ell'}^+$, we conclude that $n_1(p', q) > n_1(p', p'')$.

So, suppose that there exists a point $p \in Q' \cap H_{\ell'}^-$ such that $p \in \mathring{H}_\ell^+$. Let ℓ'' be a line extending $[p, q]$. We choose p such that the size of $H_{\ell''}^- \cap H_{\ell'}^- \cap Q'$ is maximal. Using Lemma 5, we may assume that the ray of ℓ'' emanating from q and not containing p is in H_l^-, hence does not intersect l'. It implies, considering q, that $Q'' \cap H_{l'}^+ \subsetneq H_{l''}^+$. Now suppose that there is a point x in $Q'' \cap \mathring{H}_{\ell''}^+ \cap H_{\ell'}^-$. But then, since $q \in Q'' \cap \ell''$, any line extending $[q, x]$ will contradict the choice of p, because its halfplane containing p also contains $H_{\ell''}^- \cap K'$. Hence $Q' \cap H_{\ell'}^- \subset H_{\ell''}^-$, and $n_1(p, q) > n_1(p', q')$, contradicting the choice of p' and q'.

Analogously, taking two points $q' \in Q'$ and $q'' \in Q''$ such that for some geodesic line ℓ'' extending $[q', q'']$ the value of $|Q' \cap H_{\ell''}^+| + |Q'' \cap H_{\ell''}^-| = n_2(q', q'')$ is as large as possible, we will obtain that $n_2(q', q'') = |Q'| + |Q''|$ and thus ℓ'' is an inner bitangent.

Finally, if ℓ is a third inner bitangent extending an other segment $[p, q]$ with $p \in Q'$ and $q \in Q''$, then ℓ will necessarily intersect twice one of the lines ℓ' or ℓ'. This establishes that any inner bitangent of K', K'' either extends $[p', p'']$ or $[q', q'']$. □

Two sets K', K'' of S are called *line-separable* if there exists a geodesic line ℓ such that $K' \subset \mathring{H}_\ell^-$ and $K'' \subset \mathring{H}_\ell^+$. We will call two sets K', K'' *weakly line-separable* if there exists a line ℓ such that $K' \subseteq H_\ell^-$ and $K'' \subseteq H_\ell^+$. In the first case, we will say that the line ℓ *separates* K' and K'' and in the second case that ℓ *weakly separates* K' and K''.

Lemma 15. *If (Q', Q'') is a bipartition of a finite set Q of S such that the convex hulls $K' = \mathrm{conv}(Q'), K'' = \mathrm{conv}(Q'')$ are disjoint, then K' and K'' are line-separable.*

Proof. Let ℓ' and ℓ'' be two inner bitangents of K', K'' defined as in Lemma 14: ℓ' extends $[p', p'']$ and ℓ'' extends $[q', q'']$ with $\{p', q'\} \subset Q'$ and $\{p'', q''\} \subset Q''$ (it may happen that $p' = q'$ or $p'' = q''$). Let $[p', p''] \cap [q', q''] = [u, v]$, where $u \in [p', v]$.

Case 1. $u \in K'$ and $v \in K''$.

Since K' and K'' are disjoint, $u \neq v$. Let w be an arbitrary point of $[u, v]$ not belonging to K' and K'' and let $\epsilon > 0$ be such that $B(w, \epsilon) \cap (K' \cup K'') = \varnothing$ (it exists since K', K'' are closed by Lemma 11). Let x and y be two points from different connected components of $B(w, \epsilon) \setminus [u, v]$. Let ℓ be a geodesic line extending $[x, y]$. By convexity of $B(w, \epsilon)$, $[x, y] \subset B(w, \epsilon)$ and $\varnothing \neq \ell \cap [u, v] \subset B(w, \epsilon)$. Hence the two disjoint rays of ℓ with endpoints x and y are disjoint from ℓ' and ℓ'', thus ℓ separates K' and K''.

Case 2. $u \notin K'$ or $v \notin K''$, say $v \notin K''$.

Let $\epsilon > 0$ such that $B(v, \epsilon) \cap (K' \cup K'') = \varnothing$. Let w be a point in $B(v, \epsilon/2) \setminus (\ell' \cup \ell'')$, in the same connected component as K'' in $S \setminus (\ell' \cup \ell'')$. Because $w \notin K''$, by Lemma 13 there are two geodesic $[q_1, w]$, $[w, q_2]$ with $\{q_1, q_2\} \subset Q''$, extendable in two distinct lines ℓ_1 and ℓ_2 respectively, both tangent to $\mathrm{conv}(Q'' \cup \{w\})$. Denote by H_1 and H_2 the closed halfplanes containing K'' delimited by ℓ_1 and ℓ_2 respectively. For $i \in \{1, 2\}$, the rays of ℓ_i must each intersect a distinct side of the triangle $\Delta(v, p'', q'')$. Since $\ell_i \cap [p'', q''] \subset \{p'', q''\}$, ℓ_i must intersects both $[v, p''] \subset \ell'$ and $[v, q''] \subset \ell''$, and $v \notin l_i$. Clearly $K' \subset H_i$.

Choose $x \in (\mathring{H}_1 \setminus H_2) \cap B(w, \epsilon/2)$ and $y \in (\mathring{H}_2 \setminus H_1) \cap B(w, \epsilon/2)$, clearly $[x, y] \subset B(v, \epsilon)$. Moreover $\ell_1 \cap [x, y] \neq \varnothing$ and $\ell_2 \cap [x, y] \neq \varnothing$. Then any line extending $[x, y]$ separates K' and K''. ☐

Lemma 16. *If (Q', Q'') is a bipartition of a finite set Q in general position of S such that the convex hulls $K' = \mathrm{conv}(Q'), K'' = \mathrm{conv}(Q'')$ are weakly line-separable and $(K' \cap K'') \cap Q = \varnothing$, then the sets Q' and Q'' are line-separable.*

Proof. If K' and K'' are disjoint, then the result follows from Lemma 15. So, let $K' \cap K'' \neq \varnothing$ and let ℓ_0 be a line weakly-separating K' from K''. We denote the two halfplanes delimited by ℓ_0 as H^+ and H^- such that $K' \subset H^-$ and $K'' \subset H^+$. Then $K' \cap K'' \subset \ell_0$, thus $K' \cap K'' = [p, q]$ for some points $p, q \in K' \cap K''$. Since $(K' \cap K'') \cap Q = \varnothing$, by Lemma 13 there exists an edge $[p', q']$ of K' and an edge $[p'', q'']$ of K'' such that $[p, q] = [p', q'] \cap [p'', q'']$ and $p \notin \{p', p''\}$ and $q \notin \{q', q''\}$. Let ℓ' and ℓ'' be two tangents of K' and K'' extending $[p', q']$ and $[p'', q'']$, respectively (ℓ_0 may coincide with one of these tangents). Notice that ℓ' and ℓ'' also separate K' and K'', as by convexity $\ell' \cap K'' = [p, q] = \ell'' \cap K'$. So, denote by $H^-_{\ell'}$ and $H^+_{\ell'}$ the halfplanes delimited by ℓ', and $H^-_{\ell''}$ and $H^+_{\ell''}$ the halfplanes delimited by ℓ'', such that $K' \subset H^-_{\ell'} \cap H^-_{\ell''}$ and $K'' \subset H^+_{\ell'} \cap H^+_{\ell''}$.

Let $\epsilon > 0$ be such that $\min\{d(p, p'), d(p, p''), d(q, q'), d(q, q'')\} > \epsilon$. Pick two points x and y in $\mathring{H}^+_{\ell'} \cap \mathring{H}^-_{\ell''}$ so that $x \in B(p, \epsilon)$ and $y \in B(q, \epsilon)$. Let ℓ be any geodesic line extending $[x, y]$. We assert that ℓ line-separate Q' and Q''. Since $\{x, y\} \subset \mathring{H}^+_{\ell'} \cap \mathring{H}^-_{\ell''}$, from Lemma 4 we conclude that $[x, y] \subset H^+_{\ell'} \cap H^-_{\ell''}$. Since $K' \subset H^-_{\ell'}$ and $K'' \subset H^+_{\ell''}$, we conclude that $\ell \cap K' \subset \ell' \cap K' = [p', q']$ and $\ell \cap K'' \subset \ell'' \cap K'' = [p'', q'']$. In particular, $[p, q] \subset \ell$ and

$K' \subset H_\ell^-, K'' \subset H_\ell^+$. Therefore, if ℓ does not line-separate Q' and Q'', there exists a point of Q' or Q'' on ℓ, say Q', hence on $l \cap K' \subset [p', q']$. This contradicts the general position assumption for p', q'. □

4. Crofton formula (after R. Alexander)

In this section we will prove a Crofton formula for finite subsets in general position of Busemann surfaces. Our proof uses the convexity results of previous section and follows closely and generalizes an analogous result of Alexander [1] for "planes for which the lines are the shortest paths between points".

Let (S, d) be a Busemann surface and let $Q = \{p_1, \ldots, p_n\}$ be a finite set of distinct points of S in general position. A line ℓ of S is said to *separate* the points of Q if (i) none of the points of Q lie on ℓ, and (ii) each of the open halfplanes determined by ℓ contains at least one of the points of Q. Two separating lines are called *equivalent* if each separates Q into the same pair of sets. Let $L_1, \ldots, L_m$ be the equivalence classes of lines of S separating the points of Q. For a line ℓ separating the points of Q into two nonempty subsets Q', Q'', let $K' = \text{conv}(Q')$ and $K'' = \text{conv}(Q'')$. We will say that a pair of points $p_i, p_j \in Q$ constitutes an *extremal segment* $[p_i, p_j]$ for the pair (K', K'') (or (Q', Q'')) if any line ℓ extending $[p_i, p_j]$ weakly separates K' and K''. We will call an extremal segment $[p_i, p_j]$ *positive* if either $p_i \in Q', p_j \in Q''$ or $p_i \in Q'', p_j \in Q'$ and we call $[p_i, p_j]$ *negative* if either $p_i, p_j \in Q'$ or $p_i, p_j \in Q''$.

By Lemma 14 (and similarly to the Euclidean plane) the positive extremal segments correspond to the two pairs $[p', p'']$ and $[q', q'']$ of segments with $p', q' \in K', p'', q'' \in K''$ such that any line extending one of these segments is an inner bitangent of K' and K''. On the other hand, since any line extending a negative segment of K' (or of K'') is a tangent of K' (or of K''), the negative extremal segments are edges of K' or K''. In the Euclidean plane, they are precisely the edges of K' and K'' which belongs to the triangles $\Delta^*(p', q', x)$ and $\Delta^*(p', q', x)$, where $x = [p', p''] \cap [q', q'']$. We will show now that a similar property holds for Busemann surfaces:

Lemma 17. *An edge $[p, q]$ of K' is a negative extremal segment of (K', K'') if and only if $p, q \in \Delta^*(p', q', x)$, where $x \in [p', p''] \cap [q', q'']$.*

Proof. Let ℓ', ℓ'' be two inner bitangents of K', K'' extending $[p', p'']$ and $[q', q'']$, respectively. Let A', A'', B', B'' be the four pairwise intersections of closed halfplanes defined by the lines ℓ' and ℓ'', so that $K' \subset A'$ and $K'' \subset A''$. Pick any edge $[p, q]$ of K' and let ℓ be any line extending $[p, q]$. Then ℓ is a tangent of K'. From Lemma 13, either both p, q belong to $\Delta^*(p', q', x)$ or neither of p or q belong to the interior of $\Delta^*(p', q', x)$.

First suppose that $p, q \in \Delta^*(p', q', x)$. Since ℓ is a tangent of K', ℓ does not intersect $[p', q']$. Hence, ℓ necessarily intersects the sides $[x, p']$ and $[x, q']$ of $\Delta^*(p', q', x)$ and enters in the regions B', B''. If $[p, q]$ is not an extremal segment, then ℓ intersects K'' (and therefore A''). This means that ℓ will intersect twice ℓ' or ℓ'', a contradiction. This proves that any edge $[p, q]$ of K' with $p, q \in \Delta^*(p', q', x)$ is a negative extremal segment.

Conversely, suppose that $[p,q]$ is an edge of K' such that $p,q \notin \Delta^*(p',q',x) \setminus \Delta(p',q',x)$. Suppose by way of contradiction that $[p,q]$ is a negative extremal segment, i.e., ℓ weakly separates K' from K''. Then necessarily ℓ intersects the boundary of A' consisting of rays of ℓ', ℓ'', say ℓ intersects $\ell' \cap A'$. If ℓ does not intersect the interior of B', we will conclude that ℓ passes via the point p', contrary to the assumption that the points of Q are in general position. If ℓ intersects the interior of B' (i.e., $B' \setminus (\ell' \cup \ell'')$), then since ℓ weakly separates K' and $K'' \subset A''$, ℓ necessarily intersects the second time the line ℓ', which is impossible. Hence $[p,q]$ cannot be a negative extremal segment. □

For an equivalence class L_t of lines, $t = 1, \ldots, m$, we will denote by Q'_t and Q''_t the two subsets of Q into which any line ℓ of L_t separates the points of Q and by K'_t and K''_t we denote their convex hulls. Let also ES^+_t and ES^-_t be the sets of positive and negative extremal segments of the pair (K'_t, K''_t). For an equivalence class L_t of lines, let

$$\sigma_t := \sum_{[p_i,p_j] \in ES^+_t} d(p_i,p_j) - \sum_{[p_i,p_j] \in ES^-_t} d(p_i,p_j).$$

Lemma 18. *For any equivalence class L_t of lines, $\sigma_t \geq 0$.*

Proof. Let $p', q' \in K'_t$ and $p'', q'' \in K''_t$ such that $[p',q']$ and $[p'',q'']$ are the positive extremal segments of K', K''. Let $x \in [p',p''] \cap [q',q'']$. Let

$$\sigma'_t = d(x,p') + d(x,q') - \sum \{d(p,q) : [p,q] \in ES^-_t, p,q \in K'\}$$

and

$$\sigma''_t = d(x,p'') + d(x,q'') - \sum \{d(p,q) : [p,q] \in ES^-_t, p,q \in K''\}.$$

Since $\sigma_t = \sigma'_t + \sigma''_t$, it suffices to show that $\sigma'_t \geq 0$ and $\sigma''_t \geq 0$.

To show $\sigma'_t \geq 0$ we proceed by induction on the number of negative extremal segments of K'. By Lemmata 17 and 13, these segments form a path constituted by all edges of K' located in $\Delta^*(p',q',x)$. Let p_0q_0 be a negative extremal segment with $p_0 = p'$. If $q_0 = q'$, then $[p_0,q_0]$ is the unique negative extremal segment and by triangle inequality $d(p_0,q_0) = d(p',q') \leq d(p',x) + d(x,q')$. So, let $q_0 \neq q'$. Then q_0 belongs to the interior of $\Delta^*(p',q',x)$. Let ℓ be a line extending $[p_0,q_0]$. Then ℓ necessarily intersects the segment $[x,q']$ in a point x'. All negative extremal segments of K' except $[p_0,q_0]$ are located in the triangle $\Delta^*(q_0,q',x')$. Denote their total length by σ. By induction assumption, $d(q_0,x') + d(q',x') \geq \sigma$. On the other hand, $d(p_0,q_0) + d(q_0,x') = d(p',x') \geq d(p',x) + d(x,x')$ by triangle inequality. Putting these two inequalities together and taking into account that $p' = p_0$, we obtain:

$$\begin{aligned} d(p',x) + d(x,q') &= d(p',x) + d(x,x') + d(x',q') \geq d(p',x') + d(x',q') \\ &= d(p_0,q_0) + d(q_0,x') + d(x',q') \geq d(p_0,q_0) + \sigma \\ &= \sum \{d(p,q) : [p,q] \in ES^-_t, p,q \in K'\}. \end{aligned}$$

□

Here is the main result of this section:

Proposition 1. *(Crofton formula) Let $Q = \{p_1, \ldots, p_n\}$ be a finite set of points in general position of a Busemann surface (S, d). Then for any two points $p_i, p_j \in Q$, we have*

$$(1) \quad 2d(p_i, p_j) = \sum \{\sigma_t : t \in [1, m] \text{ and any line of } L_t \text{ intersects the segment } [p_i, p_j]\}.$$

In particular, the finite metric space (Q, d) is ℓ_1-embeddable.

Proof. The proof uses double counting and closely follows the proof of Lemma 1 of [1]. We will show that each segment $[q, r]$ with $q, r \in Q$ occurs as an extremal segment in such a way that $+d(q, r)$ appears exactly the same number of times as $-d(q, r)$ on the right side of the equation (1), unless $[q, r] = [p_i, p_j]$, in which case $+d(q, r)$ appears twice. We distinguish four cases:

Case 1: The points p_i, p_j, q, r are distinct and no line ℓ containing $[q, r]$ cut the segment $[p_i, p_j]$.

We assert that in this case $\pm d(q, r)$ cannot appear on the right side of (1). Indeed, if this is not the case, then there exists an equivalence class of lines L_t such that any line of L_t intersects the segment $[p_i, p_j]$ and $[q, r]$ is an extremal segment for the couple (K'_t, K''_t). By the definition of an extremal segment, there exists a line ℓ' passing via $[q, r]$ such that the convex sets K'_t and K''_t belong to different closed halfplanes defined by ℓ'. Since q, r, p_i, p_j are pairwise distinct, the points p_i, p_j are located in different open halfplanes defined by ℓ'. But then ℓ' necessarily cuts the segment $[p_i, p_j]$, a contradiction.

Case 2: The points p_i, p_j, q, r are distinct and some line ℓ containing $[q, r]$ cuts the segment $[p_i, p_j]$.

We assert that in this case there exist precisely four equivalence classes of lines which separate p_i and p_j in such a way that $[q, r]$ is an extremal segment; moreover, $[q, r]$ occurs twice as positive and twice as negative extremal segment. We will identify these equivalence classes by giving the four pairs (K'_t, K''_t).

Since the line ℓ cuts the segment $[p_i, p_j]$, ℓ separates the set $Q - \{q, r\}$ into the necessarily nonempty sets A and B. Let $\mathring{H}_\ell^-, \mathring{H}_\ell^+$ denote the open halfplanes containing respectively the sets A and B. Here are the four possible choices for the pairs (K'_t, K''_t) :

1. $(\text{conv}(A \cup \{r\}), \text{conv}(B \cup \{q\}))$,
2. $(\text{conv}(A \cup \{q\}), \text{conv}(B \cup \{r\}))$,
3. $(\text{conv}(A), \text{conv}(B \cup \{q, r\}))$,
4. $(\text{conv}(A \cup \{q, r\}), \text{conv}(B))$.

In each of four choices we need to show that the respective partition (Q'_t, Q''_t) of Q is line-separable. Equivalently, in view of Lemma 16, it is sufficient to prove that their convex hulls K'_t, K''_t are weakly-separable and the intersection $K'_t \cap K''_t$ does not contain any point of Q. Once this is shown, we immediately conclude that in the first two pairs $[q, r]$ occurs as a positive extremal segment while in the last two pairs $[q, r]$ occurs as a negative extremal segment.

Claim 1. *Each of the pairs* $(\text{conv}(A\cup\{r\}), \text{conv}(B\cup\{q\}))$*,* $(\text{conv}(A\cup\{q\}), \text{conv}(B\cup\{r\}))$*,* $(\text{conv}(A), \text{conv}(B\cup\{q,r\}))$*, and* $(\text{conv}(A\cup\{q,r\}), \text{conv}(B))$ *of convex sets is weakly-separable and their intersections with the set* Q *are empty.*

Proof of Claim 1. Since the closed halfplanes H_ℓ^- and H_ℓ^+ are convex and $A\cup\{q,r\}\subset H_\ell^-, B\cup\{q,r\}\subset H_\ell^+$, the sets $\text{conv}(A), \text{conv}(A\cup\{r\}), \text{conv}(A\cup\{q\})$, and $\text{conv}(A\cup\{q,r\})$ are contained in H_ℓ^- while the sets $\text{conv}(B), \text{conv}(B\cup\{q\}), \text{conv}(B\cup\{r\})$, and $\text{conv}(B\cup\{q,r\})$ are contained in H_ℓ^+. Hence the four pairs of convex sets from the statement of the claim are weakly-separated by the line ℓ; thus their intersections are contained in the line ℓ.

Since the points of Q are in general position, $Q\cap\ell=\{q,r\}$. Therefore, if say $q\in\text{conv}(A)$, then since q will belong to the boundary of $\text{conv}(A)$, by Lemma 13 there exist two points $r',r''\in A\subset Q$ such that $q\in[r',r'']$, a contradiction with the fact that the points of Q are in general position. This shows that the intersection of Q with the third intersection $\text{conv}(A)\cap\text{conv}(B\cup\{q,r\})$ and the fourth intersection $\text{conv}(A\cup\{q,r\})\cap\text{conv}(B)$ are both empty. Analogously, if say $\text{conv}(A\cup\{r\})\cap\text{conv}(B\cup\{q\})\cap Q\neq\varnothing$, we can suppose without loss of generality that $q\in\text{conv}(A\cup\{r\})$. Then q will be a non-extremal boundary point of the convex hull of the set $A\cup\{r\}$. By Lemma 13, there exist two points $r',r''\in A\cup\{r\}$ such that $q\in[r',r'']$. Since $A\cup\{r\}\subset Q$ and $q\notin A\cup\{r\}$, we obtain a contradiction with the assumption that the points of Q are in general position. This completes the proof of the claim. □

Case 3: The segments $[p_i,p_j]$ and $[q,r]$ are distinct but $r=p_i$.

We assert that in this case there exist precisely two equivalence classes of lines which separate p_i and p_j in such a way that $[q,r]$ is an extremal segment; moreover, $[q,r]$ occurs once as positive and once as negative extremal segment. We will identify these equivalence classes by giving the two pairs (K'_t,K''_t). We use the notation of Case 2 except that we assume that p_j is in B and we allow A to be empty. Here are the two possible choices for the pairs (K'_t,K''_t) :

1. $(\text{conv}(A\cup\{r\}), \text{conv}(B\cup\{q\}))$,
2. $(\text{conv}(A\cup\{q,r\}), \text{conv}(B))$.

Again, as in Case 2 we have to show that the respective partitions $(A\cup\{r\}, B\cup\{q\})$ and $(A\cup\{q,r\}, B)$ of Q are line-separable. Once this is proven, the first pair leads to a positive extremal segment and the second pair to a negative extremal segment. To establish the separability property, in view of Lemma 16 it suffices to show that the pairs of convex sets $(\text{conv}(A\cup\{r\}), \text{conv}(B\cup\{q\}))$ and $(\text{conv}(A\cup\{q,r\}), \text{conv}(B))$ are weakly-separable and their intersections with the set Q are empty. As in the proof of Claim 1 we conclude that the two pairs of convex sets are weakly-separated by the line ℓ; hence their intersections are contained in ℓ. If say $q\in\text{conv}(B)$ (or $q\in\text{conv}(A\cup\{r\})$), then q belongs to the boundary of $\text{conv}(B)$ (respectively, of $\text{conv}(A\cup\{r\})$) and by Lemma 13 there exist two points $r',r''\in B$ (respectively, $r',r''\in A\cup\{r\}$) such that $q\in[r',r'']$. Since $q\neq r',r''$ because $q\notin A\cup B$, we obtain a contradiction with the assumption that the points of Q are in general position.

Case 4: The segments $[p_i, p_j]$ and $[q, r]$ coincide.

We assert that in this case there exist exactly two equivalence classes of lines which separate p_i and p_j in such a way that $[q, r]$ is an extremal segment. The sets A and B are defined as in Cases 2 and 3; here we allow that either A or B to be empty. The two possible choices for the pairs (K'_t, K''_t) are:

1. $(\mathrm{conv}(A \cup \{q\}), \mathrm{conv}(B \cup \{r\}))$,
2. $(\mathrm{conv}(A \cup \{r\}), \mathrm{conv}(B \cup \{q\}))$.

As in Cases 2 and 3 one can show that the partitions $(A\cup\{q\}, B\cup\{r\}$ and $(A\cup\{r\}), B\cup\{q\})$ are line-separable. Each pair leads to a positive extreme segment, thus to a contribution of $+d(q, r) = +d(p_i, p_j)$ to the right side of the equation (1). The Cases 1-4 show the validity of (1).

Since by Lemma 18 each $\sigma_t \geq 0, t = 1, \ldots, m$, to deduce that (Q, d) is l_1-embeddable it suffices to consider each σ_t with coefficient $\frac{1}{2}$. □

5. Proof of main results

5.1. **Proof of Theorem 1.** Let (S, d) be a Busemann surface. For a finite set Q of S, denote by $N(Q)$ the number of different collinear triplets of points of Q. Clearly, $N(Q) = 0$ if and only if the points of Q are in general position.

First we will show that for any finite set (not necessarily in general position) Q of S, the metric space (Q, d) is ℓ_1-embeddable. Let $Q = \{p_1, \ldots, p_n\}$. For a given $\epsilon > 0$, in at most $m := n^3$ steps we will define a set of points $Q_\epsilon = \{p'_1, \ldots, p'_n\}$ in general position such that $d(p_i, p'_i) < \epsilon/2$ and $|d(p_i, p_j) - d(p'_i, p'_j)| < \epsilon$ for any $p_i, p_j \in Q$. For this, setting $Q_0 := Q$, we will construct a sequence of sets $Q_1, \ldots, Q_m$ such that $N(Q) = N(Q_0) > N(Q_1) > \ldots > N(Q_{m-1}) > N(Q_m) = 0$. Each set Q_{i+1} is obtained from the set Q_i by moving a single point of Q_i at distance $< \frac{\epsilon}{2m}$. We will set $Q_\epsilon := Q_m$ and denote by p'_i the final position of the point p_i under all these movements. Since each initial point can move at most m times, $d(p_i, p'_i) \leq \epsilon/2$, thus for each pair $p_i, p_j \in Q$ we will have $|d(p_i, p_j) - d(p'_i, p'_j)| < \epsilon$.

We will describe now how from a set Q with $N(Q) > 0$ to define a new set Q_1 with $N(Q_1) < N(Q)$. Let p, q, r be three points of Q such that $q \in [p, r]$. Let $\mathcal{R}$ denote the set of pairs $\{p', r'\}$ of points of Q such that q, p', q' are not collinear. This means that for any pair $\{p', r'\} \in \mathcal{R}$, the point q does not belong to the geodesic $[p', q']$ and to the cones $C(p', q') = \{x \in S : p' \in [x, r']\}$ and $C(q', r') = \{x \in S : q' \in [p', x]$. Since the set $\mathcal{R}$ is finite, the sets $R(p', q') := C(p', q') \cup [p', q'] \cup C(q', p')$ are closed and q does not belong to any such set, we conclude that $\epsilon' := \min\{d(q, R(p', q')) : \{p', r'\} \in \mathcal{R}\} > 0$. Let $\epsilon_0 = \min\{\epsilon', \frac{\epsilon}{2m}\}$. Since q contains a neighborhood homeomorphic to a circle, there exists a direction in the neighborhood of q different from the directions on $[q, p]$ and $[q, r]$. Let q_0 be a point obtained from q by moving along that direction at distance $< \epsilon_0$ from q. Let $Q_1 := Q \setminus \{q\} \cup \{q_0\}$. We assert that $N(Q_1) < N(Q)$. From the construction, $q_0 \notin R(p', q')$ for any pair $\{p', r'\} \in \mathcal{R}$, thus q_0 cannot create new collinear triplets. On the other hand, since $q_0 \notin [p, q]$ we conclude that indeed $N(Q) > N(Q_1)$.

As a result, for each $\epsilon > 0$ we can define a set $Q_\epsilon = \{p'_1, \ldots, p'_n\}$ of points in general position such that $d(p_i, p'_i) < \epsilon$ and $|d(p_i, p_j) - d(p'_i, p'_j)| < \epsilon$. By Proposition 1, the metric spaces (Q_ϵ, d) are l_1-embeddable. On the set Q we define the metric d_ϵ by setting $d_\epsilon(p_i, p_j) = d(p'_i, p'_j)$ for any two points $p_i, p_j \in Q$, where p'_i and p'_j are the images of p_i and p_j in the set Q_ϵ. Since (Q_ϵ, d) are l_1-embeddable, the metric spaces (Q, d_ϵ) are also l_1-embeddable, whence each d_ϵ belongs to the cut cone CUT_n. Since CUT_n is closed and d_ϵ converge to d when ϵ converges to 0, we conclude that $d \in \mathrm{CUT}_n$. This establishes that the finite metric space (Q, d) is l_1-embeddable. Since all finite subspaces of (S, d) are l_1-embeddable, the metric space (S, d) is L_1-embeddable by the compactness result of [5].

5.2. **Proof of Corollary 1.** First we present an example of a Busemann graph B_n, which is not isometrically embeddable into ℓ_1. The graph B_n consists of two 3-cycles $T' = u'uu''$ and $T'' = v'vv''$ and two odd $(2n+1)$-cycles C', C'' sharing the edge uv. Let $C' \cap T' = u'u, C' \cap T'' = v'v$ and $C'' \cap T' = u''u, C'' \cap T'' = v''v$; see Fig. 3. Let X_n be the planar polygonal complex obtained by replacing the cycles C', C'' by regular $(2n+1)$-gons and the 3-cycles T', T'' by equilateral triangles. Now, if $n \geq 6$, then the angles around the vertices u and v are $\geq 2\pi$. Hence X_n is a CAT(0) complex and B_n, its 1-skeleton, is a Busemann graph. To show that a graph G is not ℓ_1-embeddable, we will use the well-known fact (which can be derived from the pentagonal inequality for L_1-spaces, see [9]) that all intervals $I(a, b)$ in ℓ-graphs are convex (i. e., if $c', c'' \in I(a, b)$ and $c \in I(c', c'')$, then $c \in I(a, b)$; the interval $I(a, b)$ consists of all vertices on shortest (a, b)-paths between a and b). Let x and y be the vertices of C' and C'' opposite to the edge uv. Then $u', u'', v', v'' \in I(x, y)$, however $u, v \in I(u', v') \setminus I(x, y)$, thus $I(x, y)$ is not convex and therefore B_n is not an ℓ_1-graph.

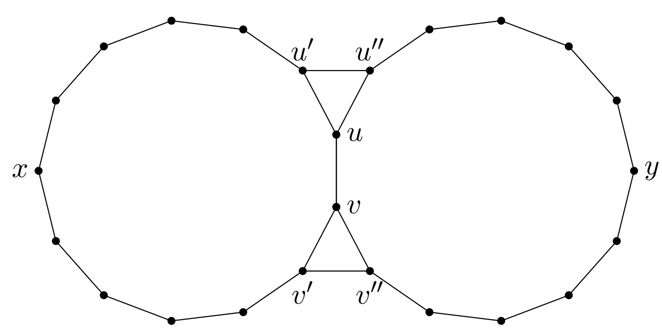

FIGURE 3. A Busemann graph that admits no isometric embedding into ℓ_1.

Now, we prove Corollary 1 that for any Busemann graph $G = (V, E)$ with a standard graph metric d_G, (V, d_G) admits an embedding into L_1 with constant distortion. Every (combinatorial) graph $G = (V, E)$ equipped with its standard distance d_G can be transformed into a (network-like) geodesic space $G' = (V', d_{G'})$ by replacing every edge $e = (u, v)$ by a segment $\gamma_{uv} = [u, v]$ of length 1; the segments may intersect only at common ends. Then (V, d_G) is isometrically embedded in a natural way in $(V', d_{G'})$. G' is often called a *metric graph*. Since G is a Busemann graph, G is the 1-skeleton of a non-positively curved regular planar complex X; let d be the intrinsic ℓ_2-metric on X. The graph G and its metric graph G' are naturally embedded in X by the identity mapping $V' \to X$.

As we noticed in Subsection 2.4, (X,d) can be extended to a Busemann surface, and this extension is an isometric embedding. Thus Theorem 1 implies that (X,d) is isometrically embeddable into L_1. Therefore, in order to show that (V,d_G) is embeddable into L_1 with distortion $c = 2+\pi/2$, it suffices to show that $(V', d_{G'})$ embeds (by the identity mapping) into (X,d) with distortion c. Pick any two points $x,y \in V'$ and let $[x,y]$ be the geodesic segment between x and y in X. Let $x =: x_0, x_1, \ldots, x_{k-1}, x_k := y$ be the consecutive intersections of $[x,y]$ with the 1-faces (edges) of X. Then each pair of consecutive points x_{i-1}, x_i belongs to a common 2-face F_i of X. Let P_i be the shortest of the two boundary paths of F_i connecting x_{i-1} and x_i and let ℓ_i be its length. Since the union $\cup_{i=1}^{k} P_i$ of these paths is a path between x and y in the metric graph G', we deduce that $d_{G'}(x,y) \leq \sum_{i=1}^{k} \ell_i$. Therefore, to prove that $d_{G'}(x,y) \leq c \cdot d(x,y)$ it suffices to show that $l_i \leq c \cdot d(x_{i-1}, x_i)$ for all $i = 1, \ldots, k$, where $d(x_{i-1}, x_i)$ is simply the Euclidean distance between two boundary points x_{i-1}, x_i of the regular polygon F_i. This is a consequence of the following result:

Lemma 19. *If a,b are two points on the boundary of a regular Euclidean polygon F and $\ell(a,b)$ is the length of the shortest boundary path P of F connecting a and b, then $\ell(a,b) \leq (2+\pi/2)d(x,y)$.*

Proof. Let C be the circle circumscribed to the regular polygon F. We distinguish two cases.

Case 1. a and b lie on incident edges $[z',z],[z,z'']$ of F. Let α be the angle between these edges. Then $d(a,b) = (d(a,z)^2 + d(b,z)^2 - 2d(a,z)d(b,z)\cos\alpha)^{\frac{1}{2}}$. Simple calculations using the fact that $\pi/3 \leq \alpha \leq \pi$ show that if $d(a,z)+d(z,b)$ is fixed, then $d(a,b)$ is minimized when α is minimized (i.e., $\alpha = \pi/3$) and $d(a,z) = d(z,b)$, in which case $d(a,b) \geq (d(a,z)+d(z,b))/2 \geq \ell(a,b)/2$.

Case 2. a and b lie on non-incident edges, say $P = (a, z_1, z_2, \ldots z_k, b)$, with $k \geq 2$. Then clearly $d(a,b) \geq 1$ and $d(a,b) \geq d(z_1,z_k)$. Also $\ell(z_1,z_k)$ (the length of the portion of P between z_1 and z_k) is upper bounded by the length of the (z_1,z_k)-arc of the circle C, and thus is at most $\frac{\pi}{2} d(z_1,z_k)$. From this we get

$$\ell(a,b) \leq 2 + \ell(z_1,z_k) \leq 2d(a,b) + \frac{\pi}{2} d(z_1,z_k) \leq \left(2+\frac{\pi}{2}\right) d(a,b).$$

□

Hence, the identity mapping $V' \to X$ of G and G' into X is a non-expansive embedding with distortion $c = 2+\pi/2$: for any two vertices x,y of G (and, more generally, for any two points $x,y \in V'$ of G') we have $\frac{1}{c} d_{G'}(x,y) \leq d(x,y) \leq d_{G'}(x,y)$. This concludes the proof of Corollary 1.

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