Polyhedral Combinatorics linking Hamiltonicity, Scattering sets and multiway cuts.

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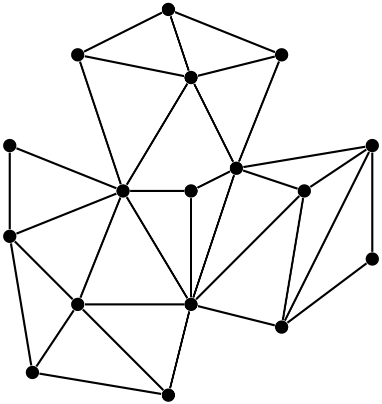
The shortest tour problem



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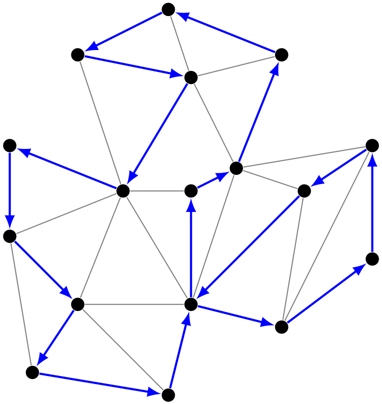

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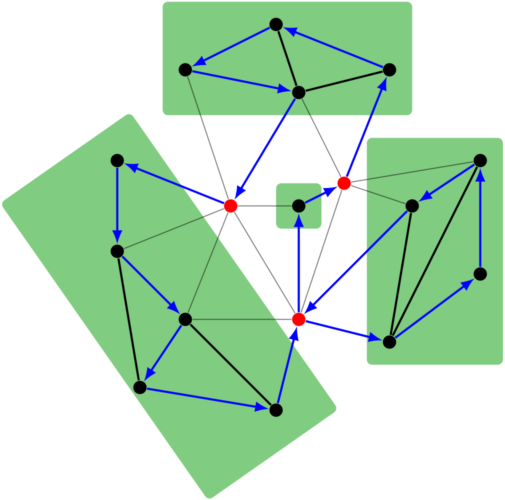


Two problems?

Definition (tour)

Closed walk spanning the vertices of a graph.

Let $c(U)$ denote the number of component of $G - U$.

Definition (scattering set)

Set U with $c(U) > |U|$.

Problem (Shortest tour)

Given a graph G, find a shortest tour of G.

Problem

Given a graph G, find a scattering set of G, if there is one.

Remark:

Definition (Toughness)

Toughness of a graph G:
minimum of $\frac{|U|}{c(U)}$ over all $U \subset V$ with $c(U) \geq 2$.

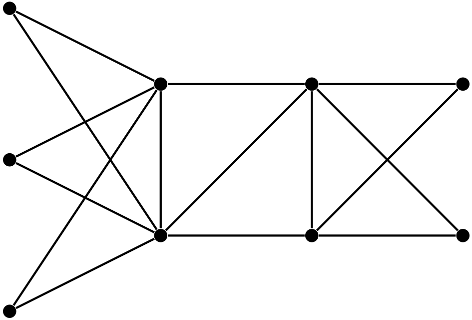
Notion introduced by Chvátal (1973)

Conjecture

There is a constant t_0 such that any t_0-tough graph is Hamiltonian.

If t_0 exists, then $t_0 \geq \frac{9}{4}$.

A more complicated example.



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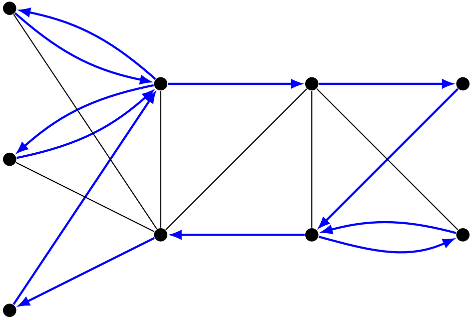

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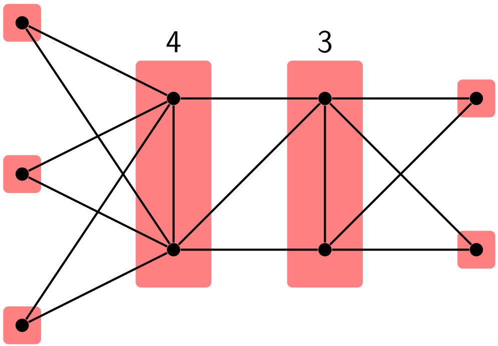

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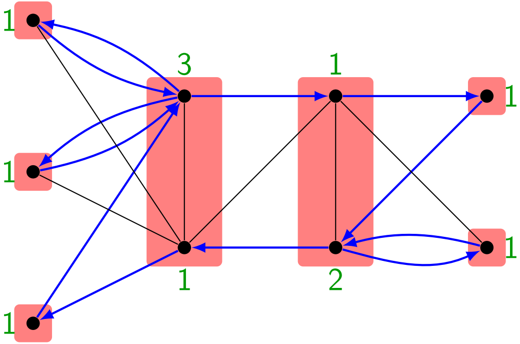

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More interesting problems.

Problem (Partition into scattering sets)

Find a partition $\mathcal{P}$ of V maximizing:

$$\sum_{U \in \mathcal{P}} c(U)$$

Problem (Covering the scattering sets)

Find a vector $x : V \to \mathbb{N}$ such that:

$$x(U) \geq c(U) \qquad (\textit{for all } U \subset V)$$

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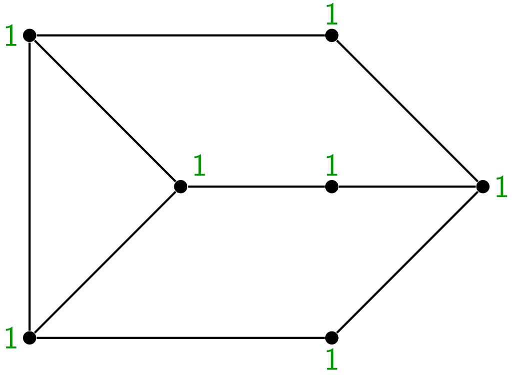
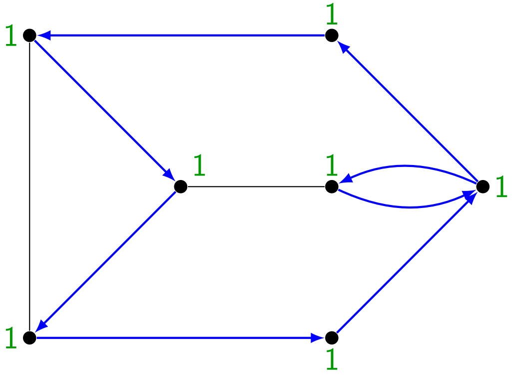
Shortest tour $\geq$ x minimum $\geq$ maximum partition

Min Cover $\neq$ Shortest Tour



Minimum Cover : 7

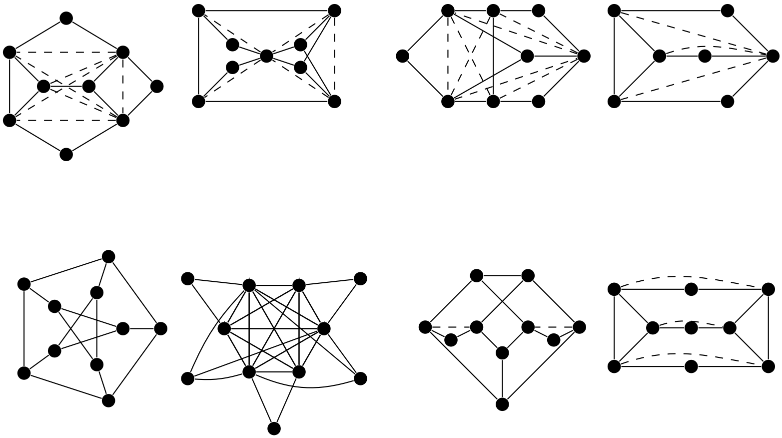
Min Cover ≠ Shortest Tour



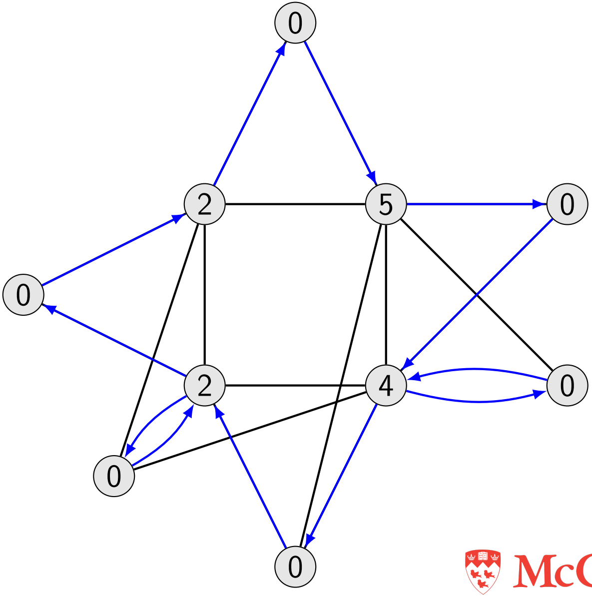
Minimum Cover : 7

Shortest Tour : 8

Bad graphs

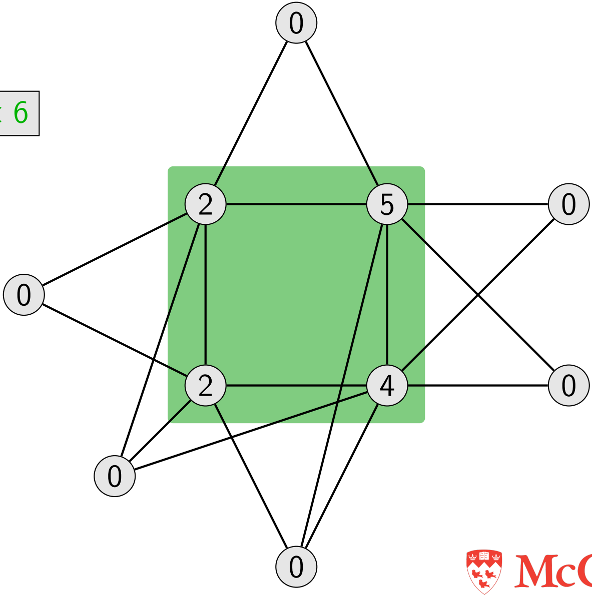


A weighted example



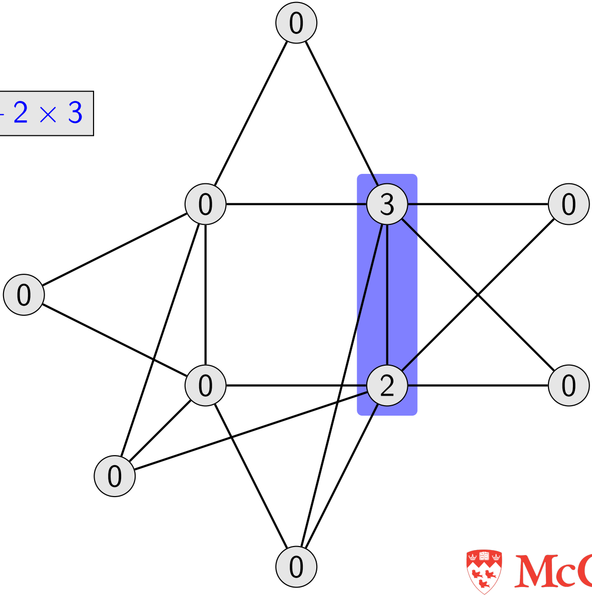
A weighted example

2×6



A weighted example

$2 \times 6 + 2 \times 3$



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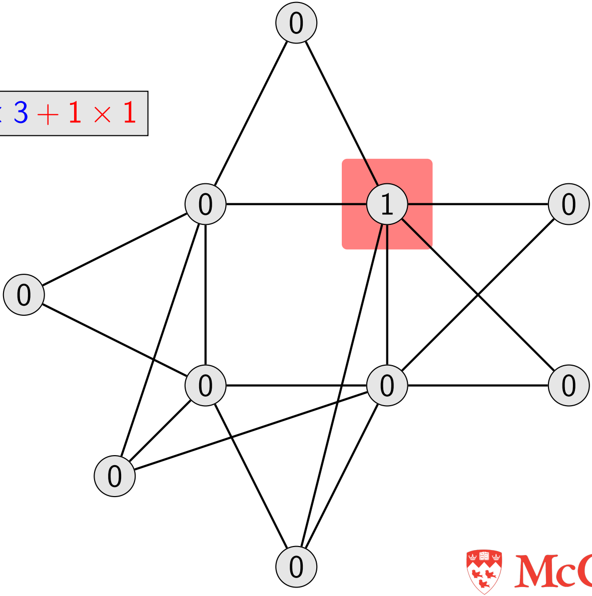
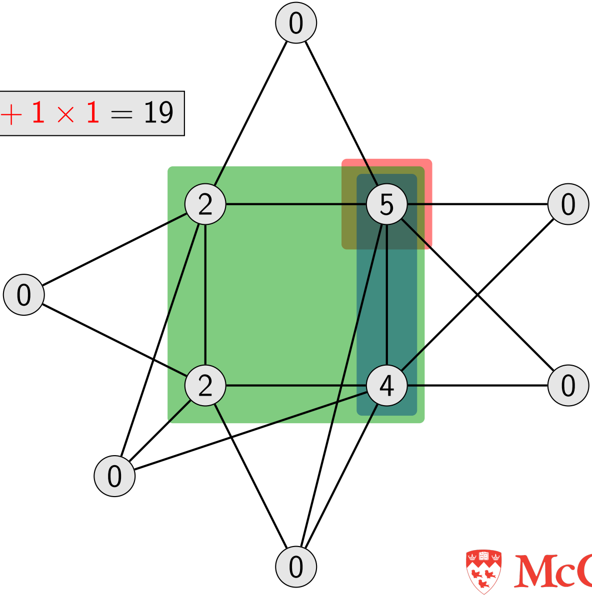
A weighted example

$2 \times 6 + 2 \times 3 + 1 \times 1$

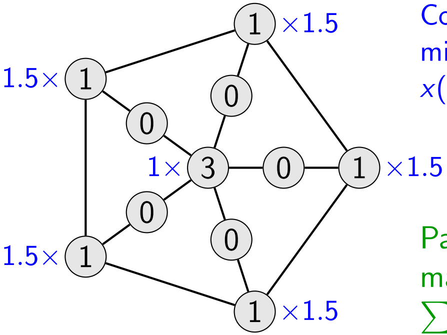


A weighted example

$2 \times 6 + 2 \times 3 + 1 \times 1 = 19$



Min Cover $\neq$ Max packing



Cover :

min wx

$x(U) \geq c(U) \qquad (U \subseteq V)$

Packing :

max $\sum_{U \subseteq V} y_U c(U)$

$\sum_{U \ni v} y_U = w_v \qquad (x \in V)$

Fractional packing : $5 \times \frac{1}{2} \times 4 \quad + \frac{1}{2} \times 1 = 10.5$

Fractional cover : $5 \times 1.5 + 3 = 10.5$

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Objective

Find classes of graphs for which:

Shortest Tour = Min Cover = Max Packing

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Theorem

True for trees.

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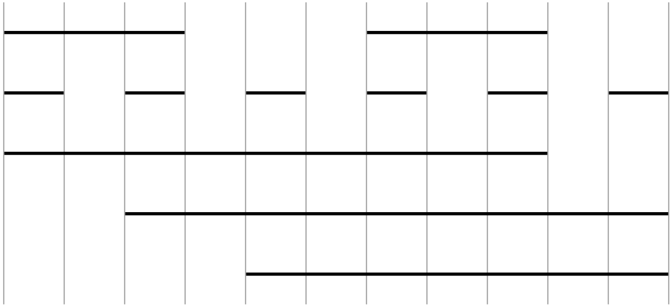
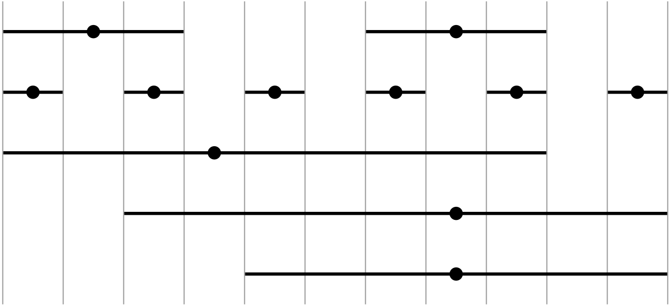
Theorem

True for cographs (induced P_4-free graphs).

Definition: interval graphs

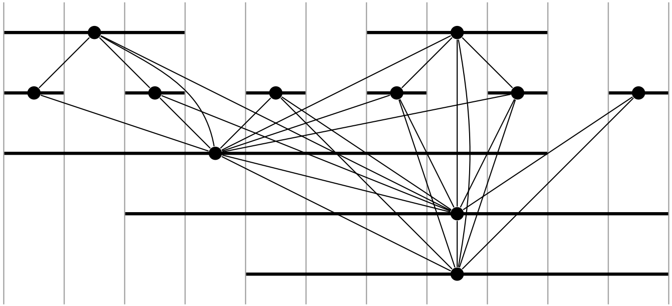


Definition: interval graphs



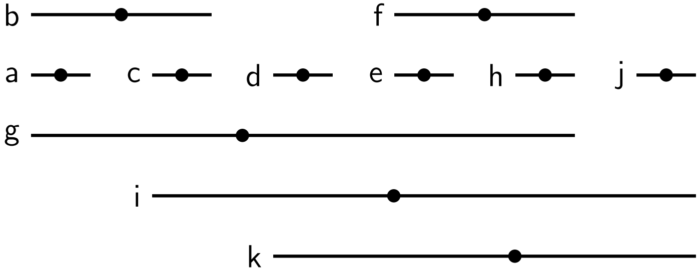
Vertices: intervals.

Definition: interval graphs



Edges: two intersecting intervals are adjacent.

Hamiltonicity in interval graphs (Keil, 1985)



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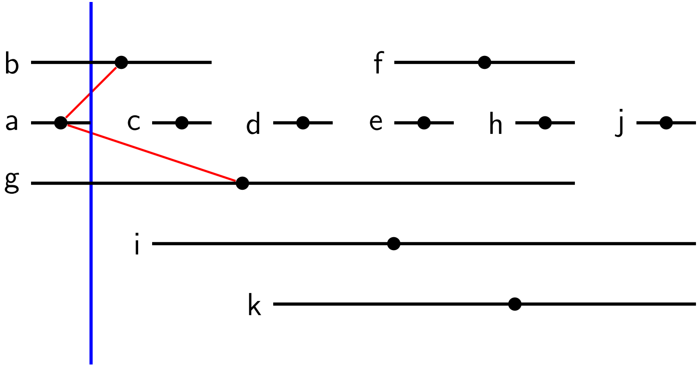

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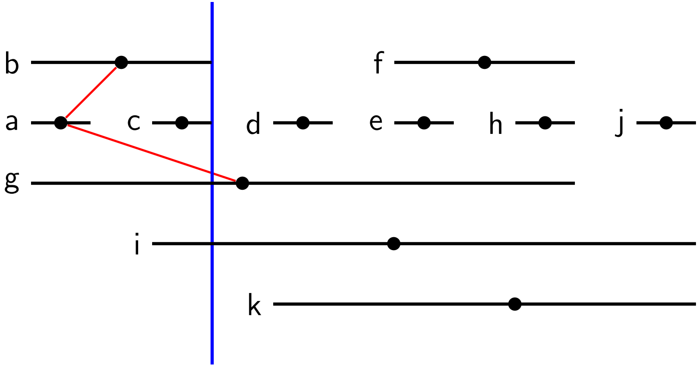

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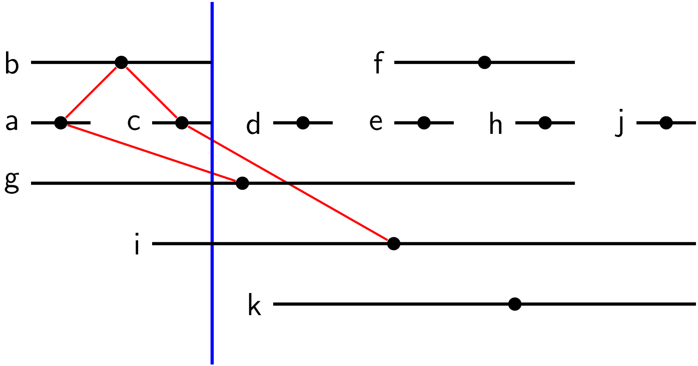

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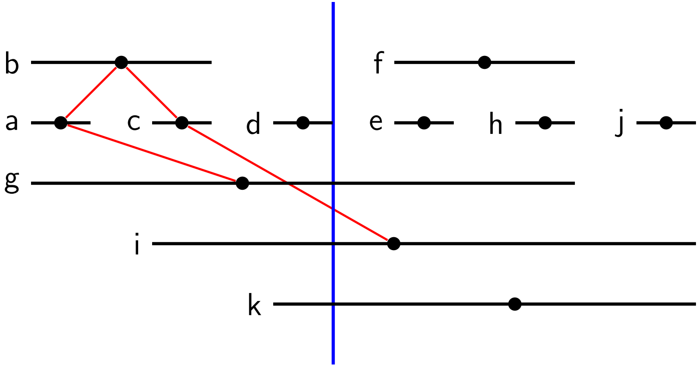

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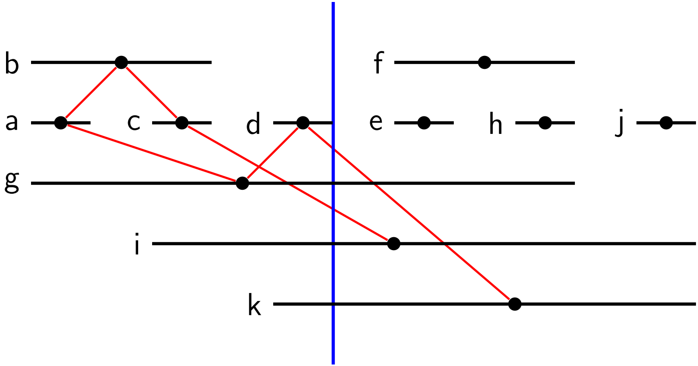

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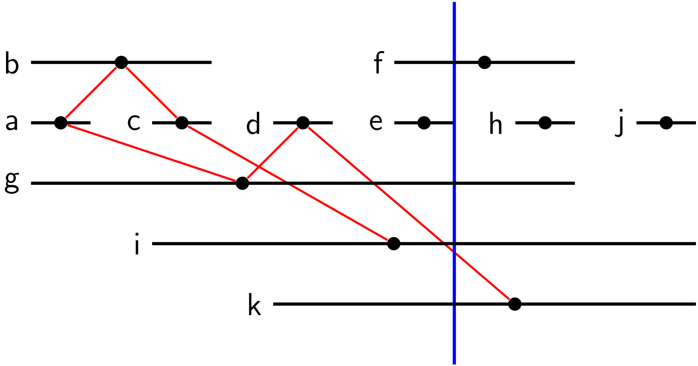

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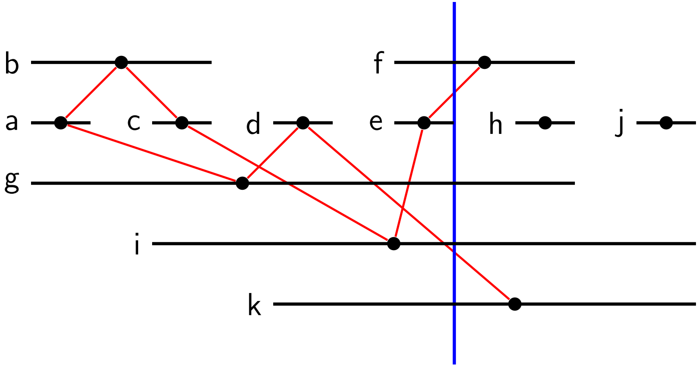

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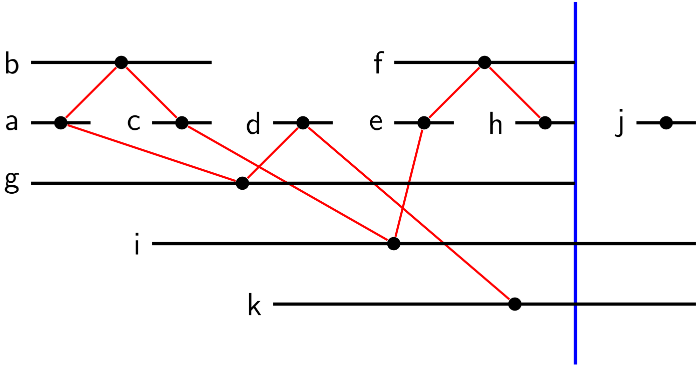

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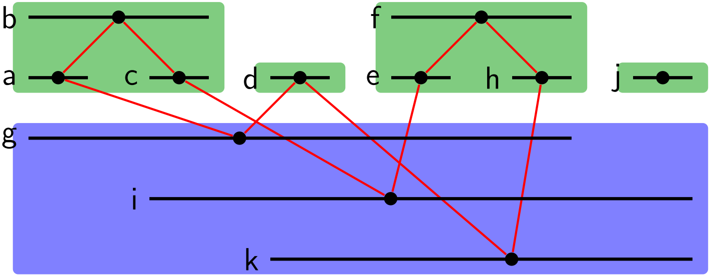
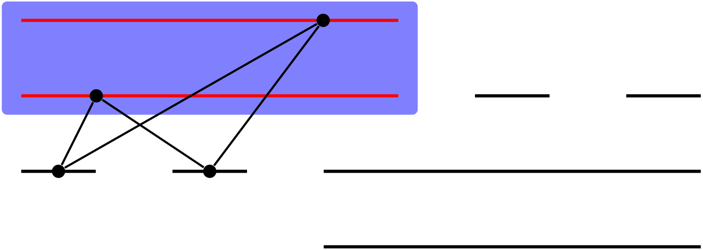
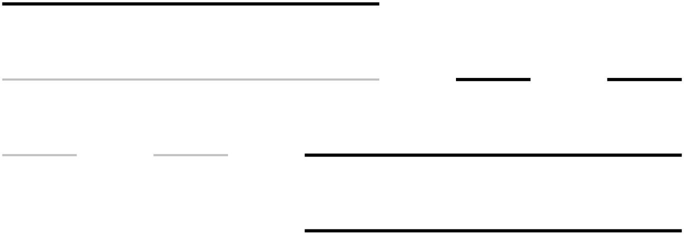


Tour and packing on interval graphs (I)



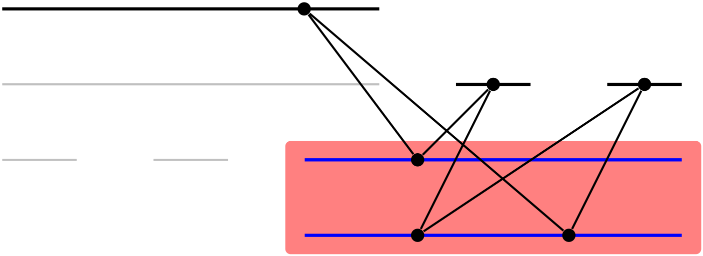
Find a left-most cycle.

Tour and packing on interval graphs (I)



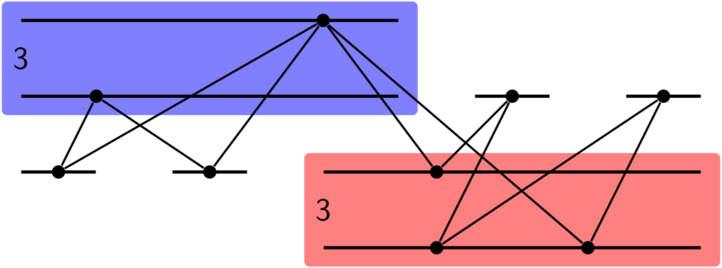
Keep the right-most interval among the subtour.

Tour and packing on interval graphs (I)



Recurse to find an optimal tour.

Tour and packing on interval graphs (I)



The two subtours are joined by their common node.

Tour and packing on interval graphs (II)

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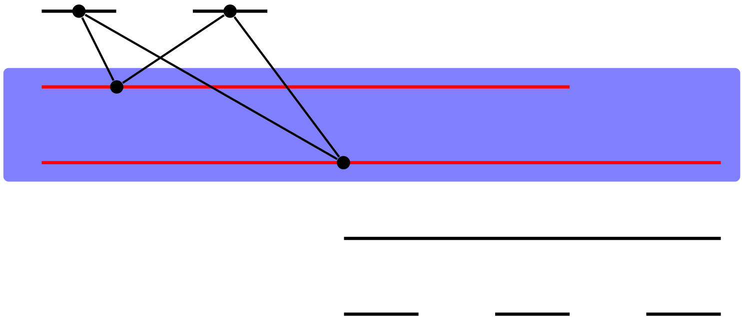

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Tour and packing on interval graphs (II)

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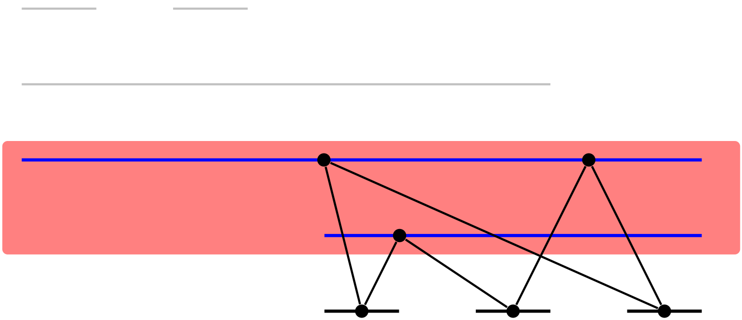
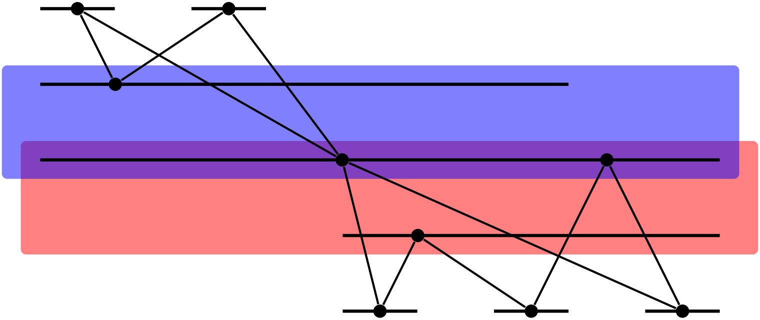
Tour and packing on interval graphs (II)



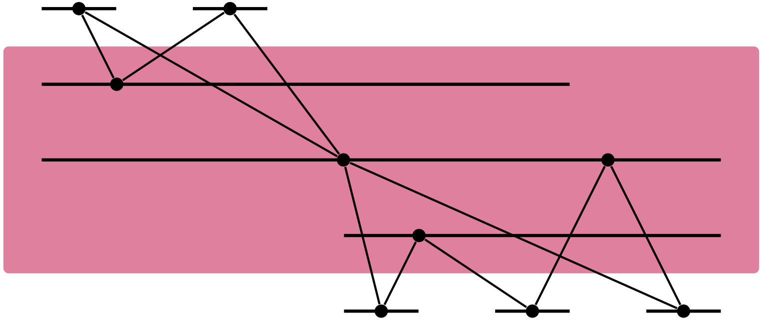
Recurse to find an optimal tour.

Tour and packing on interval graphs (II)



The two sets are not disjoint!

Tour and packing on interval graphs (II)



Take the union of the two sets.

Result

Theorem

If G is an interval graph, the cardinality of a shortest tour is equal to the maximum value of a partition into scattering sets.

Result

Theorem

If G is an interval graph, the cardinality of a shortest tour is equal to the maximum value of a partition into scattering sets.

Weighted case?

It would be enough to show that $x(U) \geq c(U)$ is TDI.

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Other classes of graphs

Cocomparability graphs : complement graphs of transitively orientable graphs.
AT-free graphs: Graphs with no induced asteroidal triple.

Interval graphs $\subset$ Cocomp $\subset$ AT-free

Conjecture: shortest tour = max partition on these classes.

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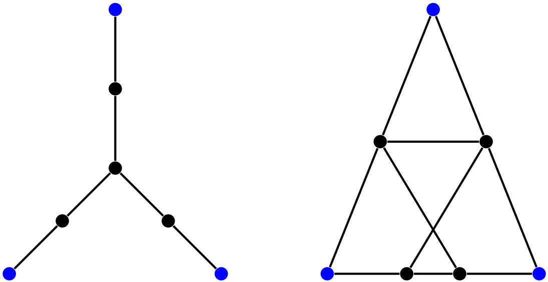
Asteroidal Triple

Graphs with no induced Asteroidal Triple (*AT-free*) :
AT : x_1, x_2, x_3 independant such that for all $\{i, j, k\} = \{1, 2, 3\}$, there is a (x_i, x_j)-path in $G - N(x_k)$.

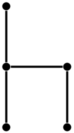
- Hamiltonicity is open on AT-free graphs.
- All our counter-examples have AT.



chair-free graphs

Theorem

The system $x(U) \geq c(U)$ (for all U) is TDI for chair-free graphs.



Does not prove: Shortest Tour = Max Packing!

Chair-free graphs (I)

Lemma (Folklore)

For all $A, B \subset V$,

$$c(A) + c(B) \leq c(A \cup B) + c(A \cap B) + d(A, B)$$

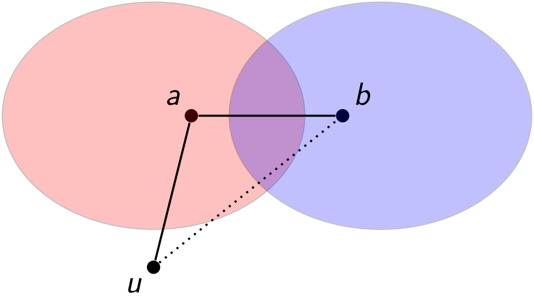
Lemma

There is an optimal packing such that for every active set A with $|A| \geq 2$, every $v \in A$ is adjacent to at least three connected components of $G - A$.

More generally: $A' \subset A$ is adjacent to $|A'| + 2$ components

Chair-free graphs (II)

- $ab \in \delta(A, B)$, $u \in \overline{A \cup B}$, $au \in E$, $bu \notin E$.



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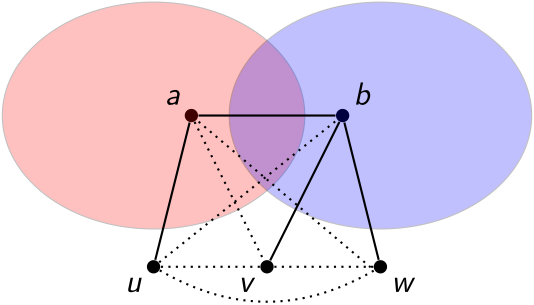

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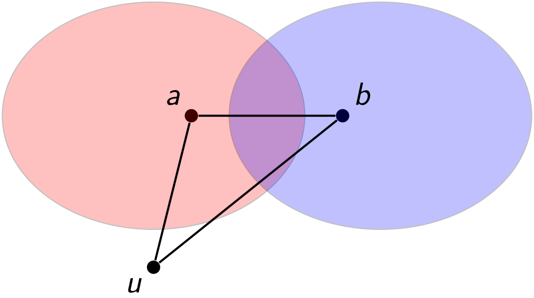

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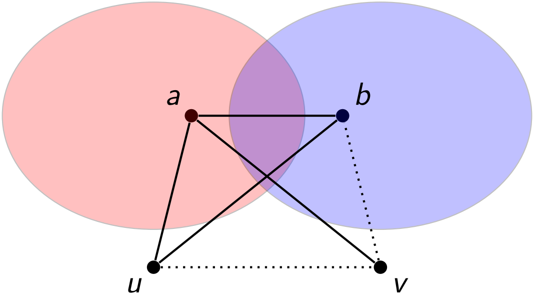

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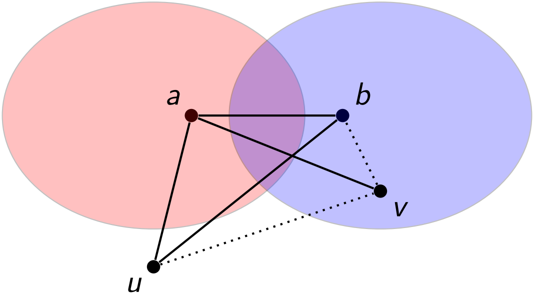

Separation of the primal

Primal LP :

$$\begin{array}{ll} \min & wx \\ x(U) \geq \mathrm{c}(U) & (U \subseteq V) \end{array}$$

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Separation:

$$|S| - x(U) > 0$$

where: S is an independant set,
U blocks the S-paths,
U and S are disjoint.

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Separation of the primal

Primal LP :

$$\begin{array}{ll} \min & wx \\ x(U) \geq \mathrm{c}(U) & (U \subseteq V) \end{array}$$

More generally:

$$\max_{S,U} y(S) - x(U)$$

where: S is an independant set,
U blocks the S-paths,
U and S are disjoint.

Special cases

- Max Independant Set ($x = 0$),

$$\max y(S) - x(U)$$

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$$\max y(S)$$

where: S is an independant set.

Special cases

- Max Independant Set ($x = 0$),
- Toughness ($y = k$, $x = 1$),

$$\max y(S) - x(U)$$

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Special cases

- Max Independant Set ($x = 0$),
- Toughness ($y = k$, $x = 1$),

$$
\begin{aligned}
&\max k|S| - |U| \\
\text{where:} \quad & S \text{ is an independant set,} \\
& U \text{ blocks the } S\text{-paths,} \\
& U \text{ and } S \text{ are disjoint.}
\end{aligned}
$$

Special cases

- Max Independant Set ($x = 0$),
- Toughness ($y = k$, $x = 1$),
- Multiway cuts ($y \in \{0; +\infty\}$).

$$\begin{aligned} & \max y(S) - x(U) \\ \text{where:} \quad & S \text{ is an independant set,} \\ & U \text{ blocks the } S\text{-paths,} \\ & U \text{ and } S \text{ are disjoint.} \end{aligned}$$

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Special cases

- Max Independant Set ($x = 0$),
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- Multiway cuts ($y \in \{0; +\infty\}$).

$$\min x(U)$$

where: $\mathcal{S}$ is an independant set,
U blocks the $\mathcal{S}$-paths,
$U \subseteq V \setminus \mathcal{S}$

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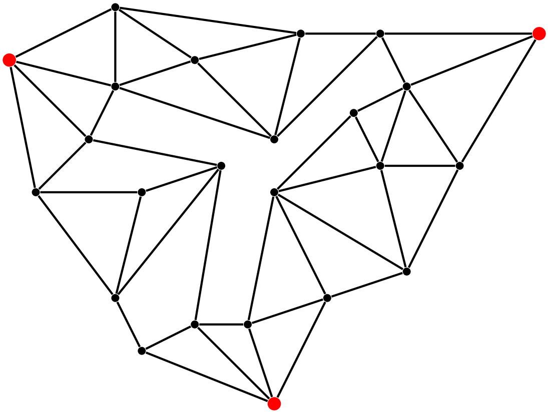
Multiway cuts and disjoint $\mathcal{S}$-paths



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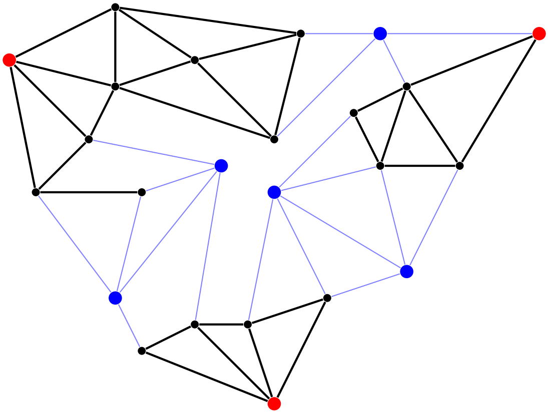

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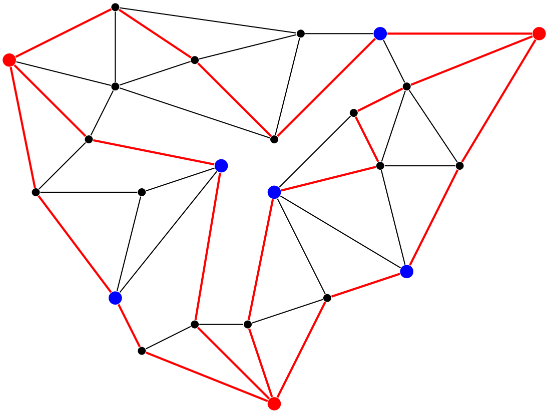
Multiway cuts and disjoint $\mathcal{S}$-paths



Two more problems...

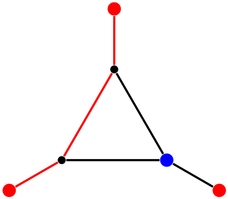
Problem (Min Blocker of T-paths)

Given a graph G, weights $w : V \to \mathbb{N}^+$, and an independant set T, find a subset of vertices S of minimum weight such that each T-path intersects S.

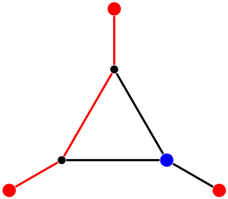
Problem (Max disjoint T-paths)

Given a graph G, weights $w : V \to \mathbb{N}^+$, and an independant set T, find a maximum packing of T-paths

. . . No Min-Max theorem.



. . . No Min-Max theorem.



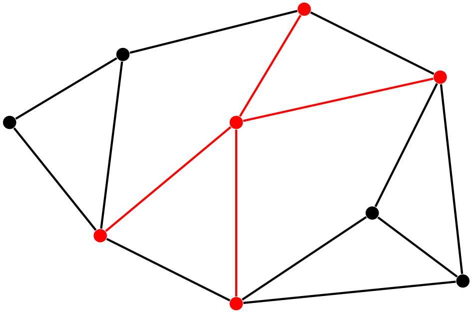
Theorem (Mader, 1978)

Good characterization for the maximum number of T-paths.

Vertex-minors

(not the usual definition of vertex-minors)
We define two operations:

- *vertex deletion*,
- *vertex contraction*.



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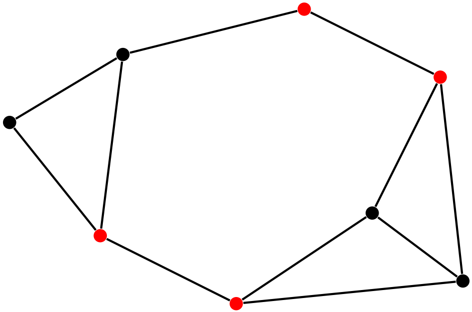

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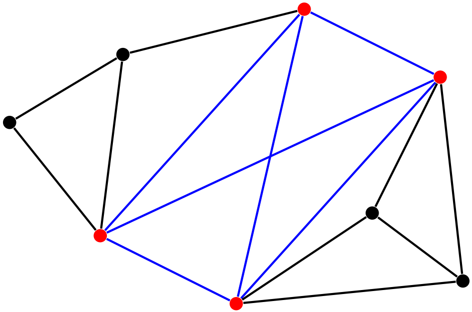
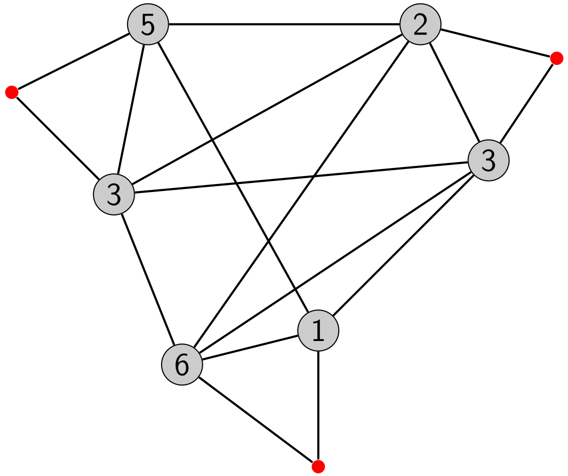


Why these operations?



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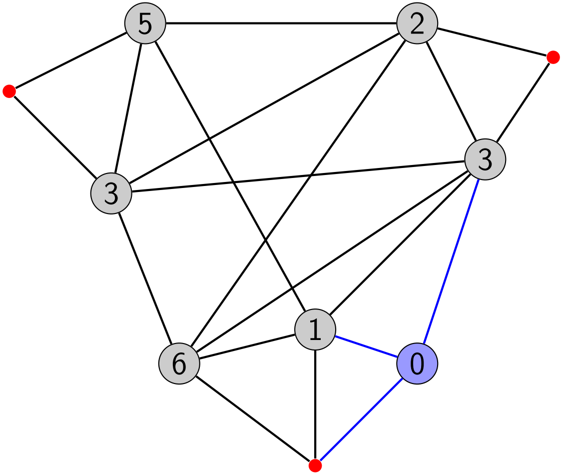

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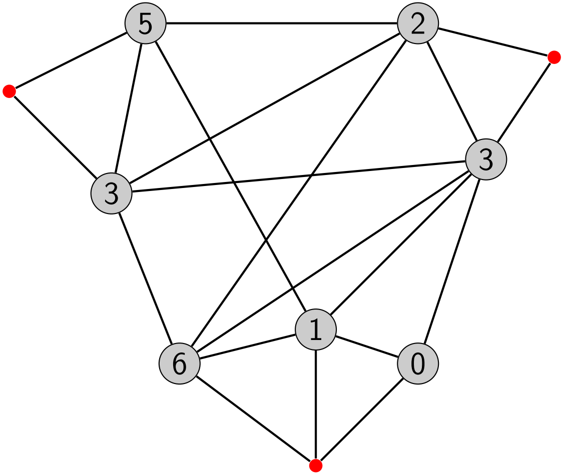

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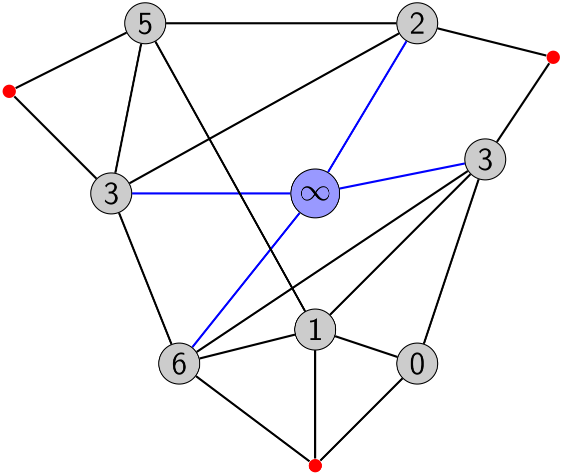

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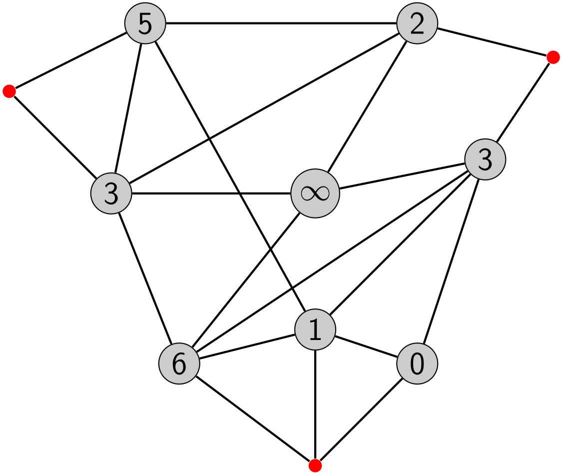


Immediate consequence

Proposition

The class of graphs for which the following system is TDI:

$$x(P) \geq 1 \quad \textit{for all } T\textit{-path } P$$

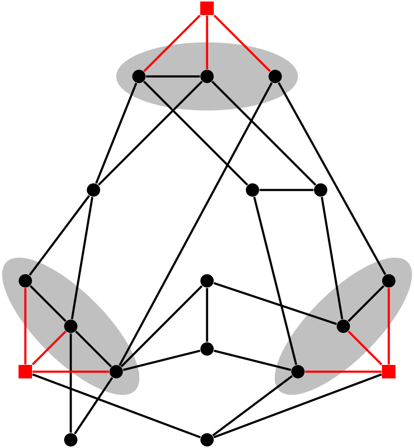
is closed under vertex-minor operations.

Vertex minors are also relevant for Steiner variant of the shortest tour problem.

TDIness of $x(P) \geq 1$

When is the following system TDI?

$$x(P) \geq 1 \quad \text{for all } T\text{-path } P$$
$$x \in \mathbb{R}_+^V$$



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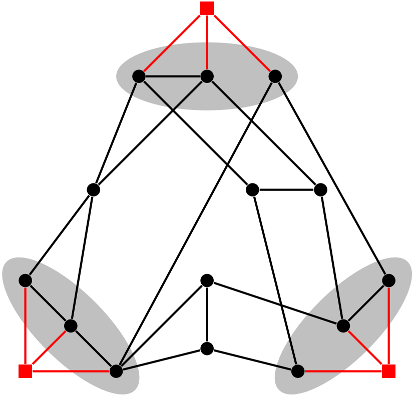

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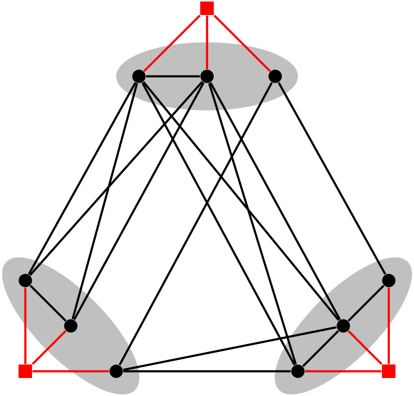

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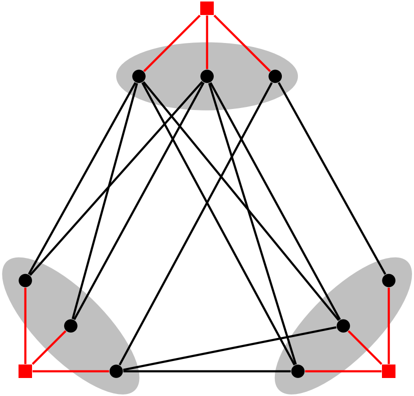

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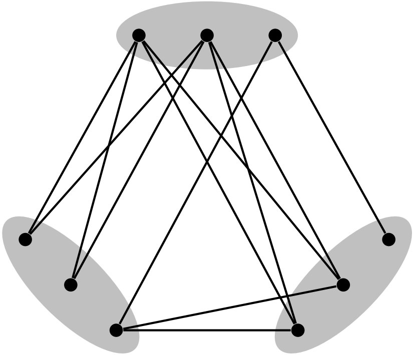

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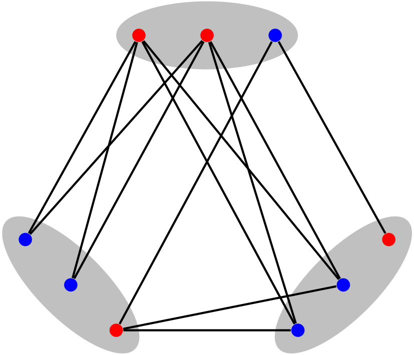

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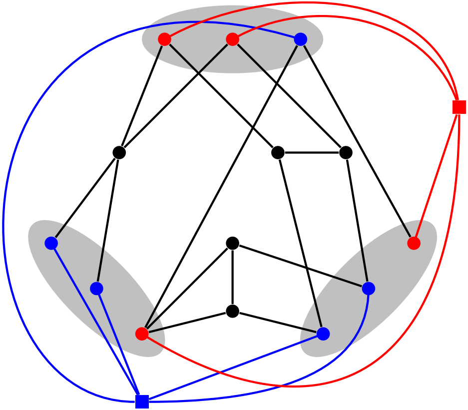

Algorithm

1. Remove nodes adjacent to two terminals.
2. Remove components of $G - (T \cup N(T))$ adjacent to only $N(t)$ for some $t \in T$.
3. Contract the other components.
4. Remove the edges in $N(t)$ for all $t \in T$.
5. If the graph obtained is bipartite, use Menger theorem to find a maximum packing of paths and a minimum multiway cut.
6. If the graph is not bipartite, the system is not TDI.

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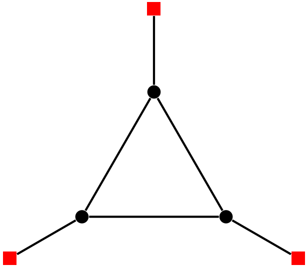
Non-bipartite auxiliary graph.



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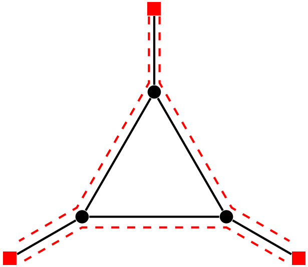

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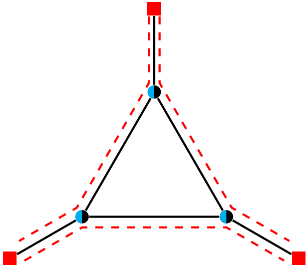

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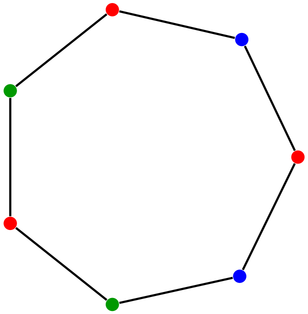

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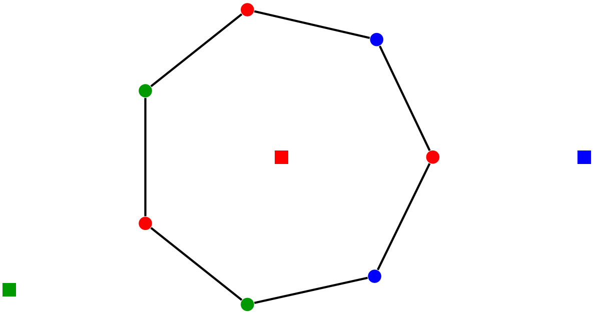

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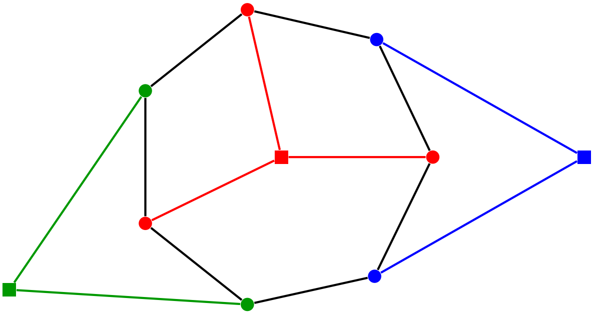

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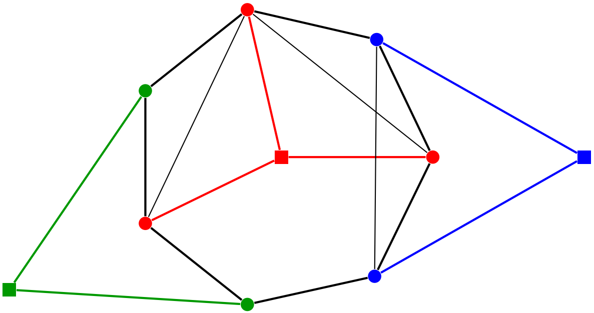

Characterization

Theorem

Let G be a graph. Exactly one is true:

- *the system $x(P) \geq 1$ for every T-path is TDI for any independant set T,*
- *or G contains a bad vertex minor.*

Moreover, in the first case:

algorithm based on min flow to solve
multiway cut and path packing.

Mader-Mengerian graphs

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Recognizing Mader-Mengerian graphs

Checking bipartiteness for any T?

Recognizing Mader-Mengerian graphs

Checking bipartiteness for any T?

Theorem

A graph is Mader-Mengerian iff its auxiliary graph is bipartite for every independant set T with $|T| = 3$.

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Recognizing Mader-Mengerian graphs

Checking bipartiteness for any T?

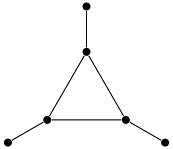
Theorem

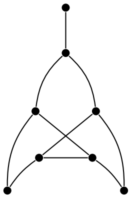
A graph is Mader-Mengerian iff its auxiliary graph is bipartite for every independant set T with $|T| = 3$.

Polynomial algorithm for recognition.

More about Mader-Mengerian graphs

- AT-free graphs are Mader-Mengerian,
- Infinite family of forbidden vertex-minors,
- graphs without rockets and nets by vertex-minor + edge-contraction are Mader-Mengerian:





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Conclusion

Done	Open
Min Tour $=$ Max Packing in interval graphs	weighted case? Extension to AT-free graphs
Covering system TDI in chair-free	TDIness in interval graphs?
Generalized separation in interval graphs	Extension to bigger class?
Mader-Mengerian graphs	?

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Thank you!