Disjoint paths in acyclic planar graphs

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The disjoint paths problem

Problem (Disjoint-Paths)

Given:

- *a supply graph G (multiple edges or arcs allowed),*
- *commodities (demands) $s_1t_1, \ldots, s_kt_k$.*

Decide whether there are disjoint paths $P_1, \ldots, P_k$, such that P_i is from s_i to t_i.

Different flavors:

- Vertex-disjoint,
- Edge-disjoint,
- Arc-disjoint.

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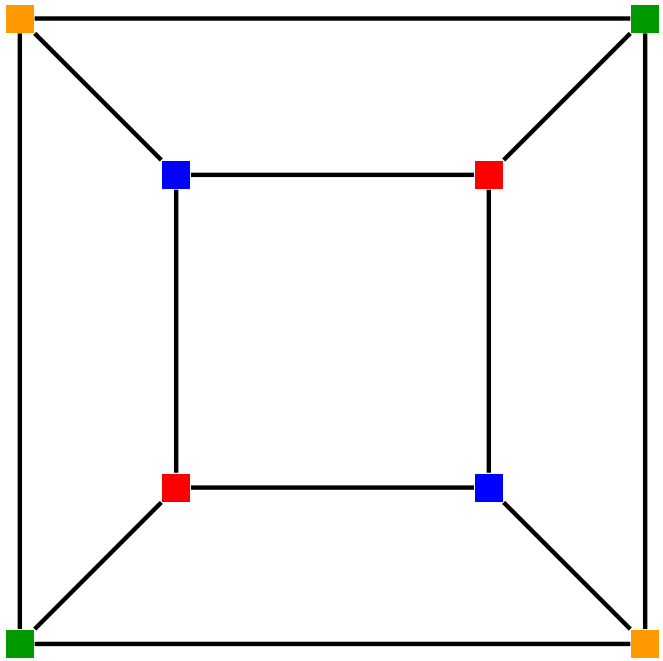
Example



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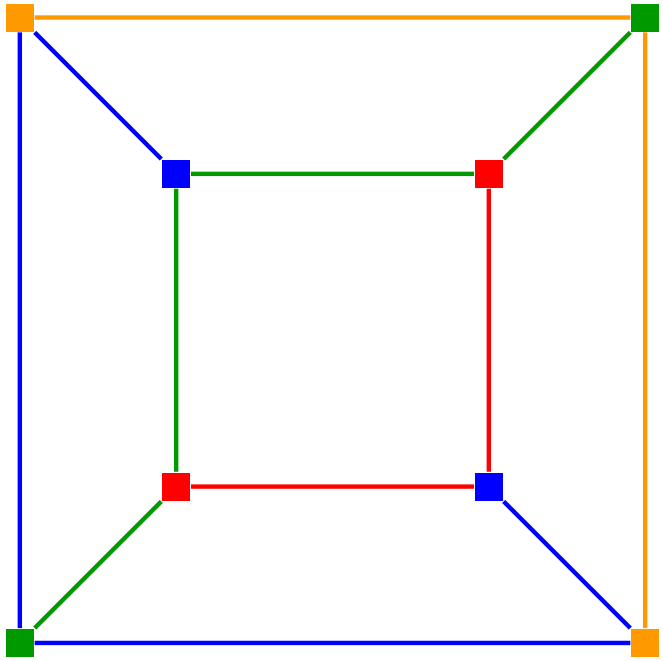

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Reformulation

Definition (Demand graph)

$E(H) := \{t_i s_i \ : \ i \in [\![1, k]\!]\}$

Encode parallel arcs by capacities:

- $c : E(G) \to \mathbb{N}$ *capacities*
- $r : E(G) \to \mathbb{N}$ *requests*

Problem (Integer-Multicommodity-Flow)

Given (G, c, H, r)*, find concurrent integral flows in* (G, c) *of values* $r(h)$ *for each* $h \in E(H)$*.*

More general!

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Eulerian condition

(directed case)

$$\sum_{e \in \delta_G^- v} c(e) + \sum_{h \in \delta_H^-(v)} r(h) = \sum_{e \in \delta_G^+(v)} c(e) + \sum_{h \in \delta_H^+(v)} r(h)$$

More generally, for any cut $X \subset V(G)$:

$$\sum_{e \in \delta_G^- X} c(e) + \sum_{h \in \delta_H^-(X)} r(h) = \sum_{e \in \delta_G^+(X)} c(e) + \sum_{h \in \delta_H^+(X)} r(h)$$

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Example of application

Theorem (Hu (undir.), Nash-Williams (dir.))

2-INTEGER-MULTIFLOW *with* $G + H$ *Eulerian is polynomial.*

Proof for directed case:

Let $E(H) = \{h_1, h_2\}$.

- If there is no h_1-flow of value $r(h_1)$, no solution,
- Otherwise, choose a flow f_1,
- $f_2 = c - f_1$ is a h_2-flow of value r_2.

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Eulerian problem

Problem

INTEGER-MULTIFLOW *in digraphs with* $G + H$ *Eulerian and* $|E(H)|$ *fixed.*

Remarks :

- If $|E(H)| = 1$ (flow in a Eulerian digraph), there always exists a flow.
- If $|E(H)| = 2$, it is polynomial (Nash-Williams, Frank).
- If $|E(H)| \geq 3$, NP-complete even if G is acyclic (Vygen).

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Planar Eulerian problem

We need to add more constraints:

- $G + H$ Eulerian,
- G planar,
- G acyclic.

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Theorem (Pfeiffer, Marx)

Arc-Disjoint-Paths *is still* NP*-complete under these restrictions.*

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Planar Eulerian problem

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- $G + H$ Eulerian,
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Bound the number of demands: $|E(H)| \leq k$?

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Is it trivial?

Planar, Eulerian, acyclic, bounded demands,...

With so many constraints, is it still interesting?

Question

In particular, does the existence of a fractional solution implies the existence of an integral solution?

Theorem (N., Weibel 2010)

Yes for k-Arc-Disjoint-Paths *with* $k \leq 5$, *No for* $k \geq 6$.

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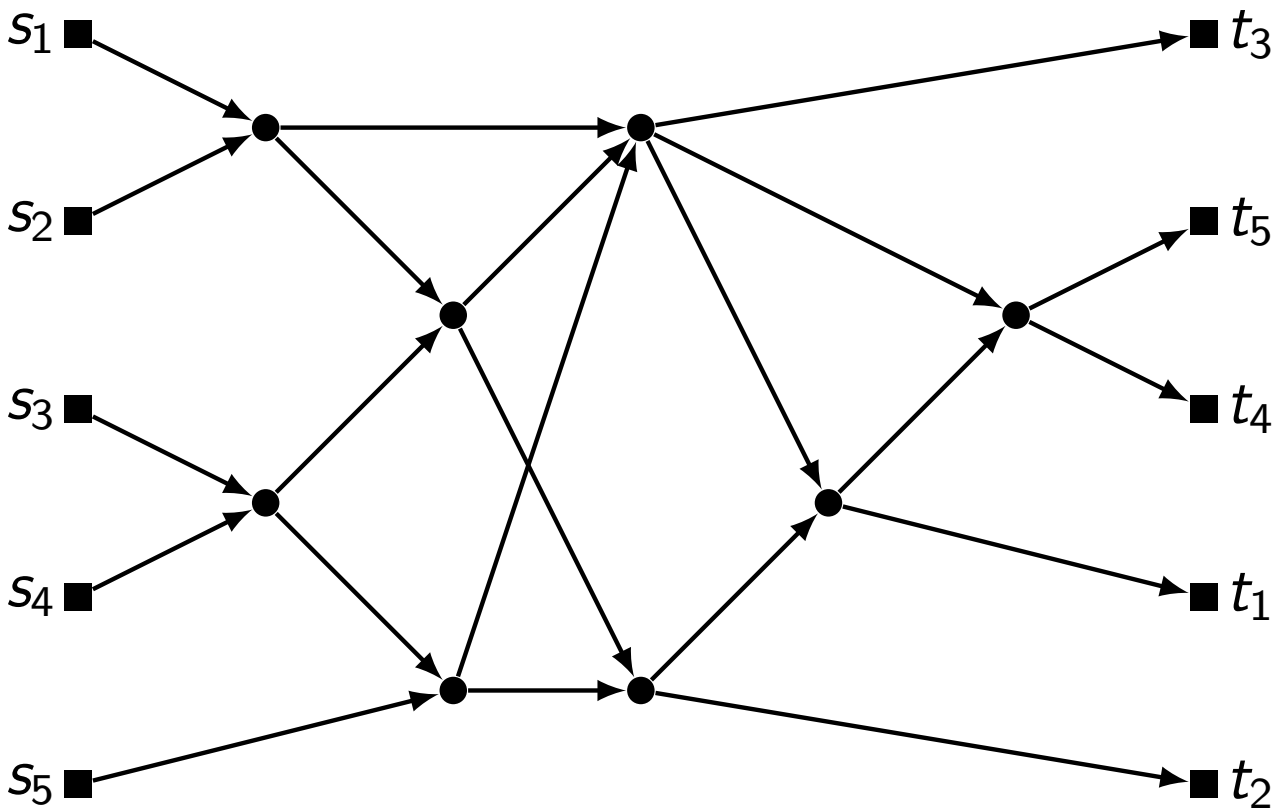
Structure of our graphs



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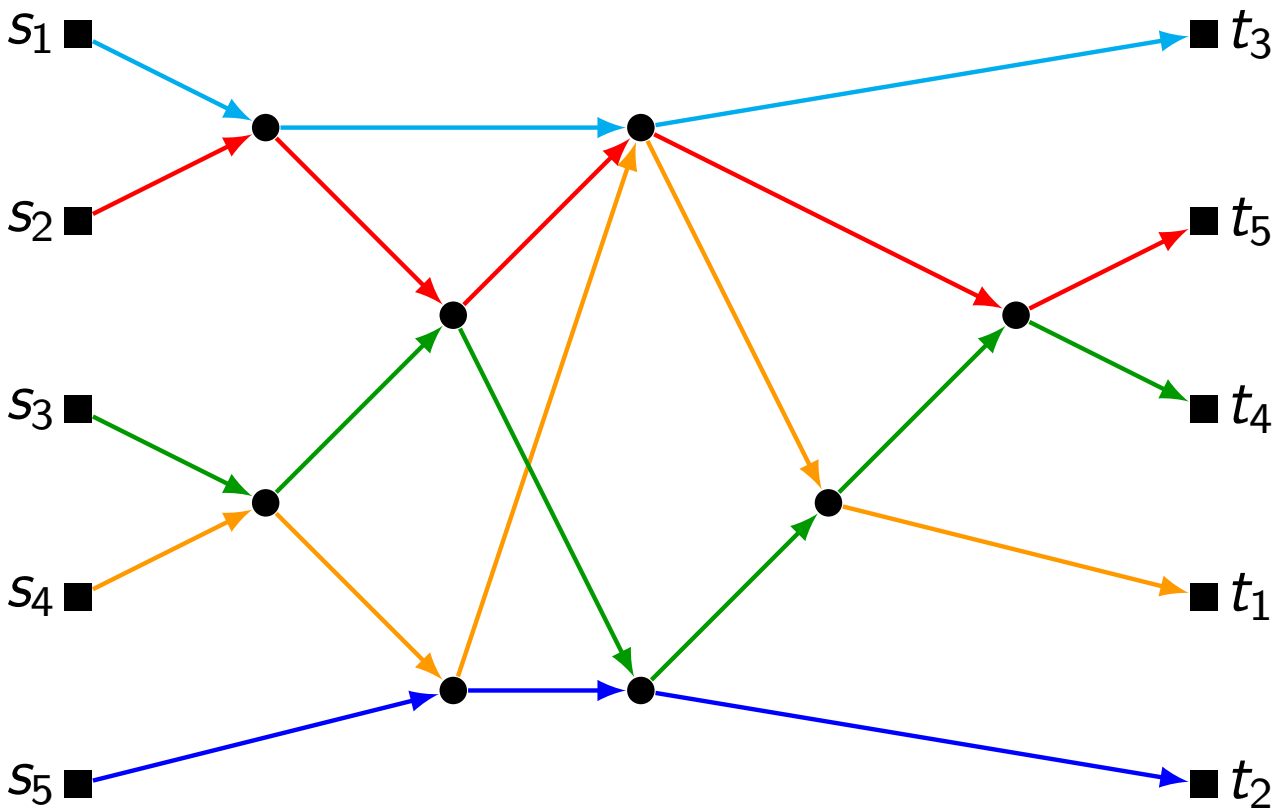

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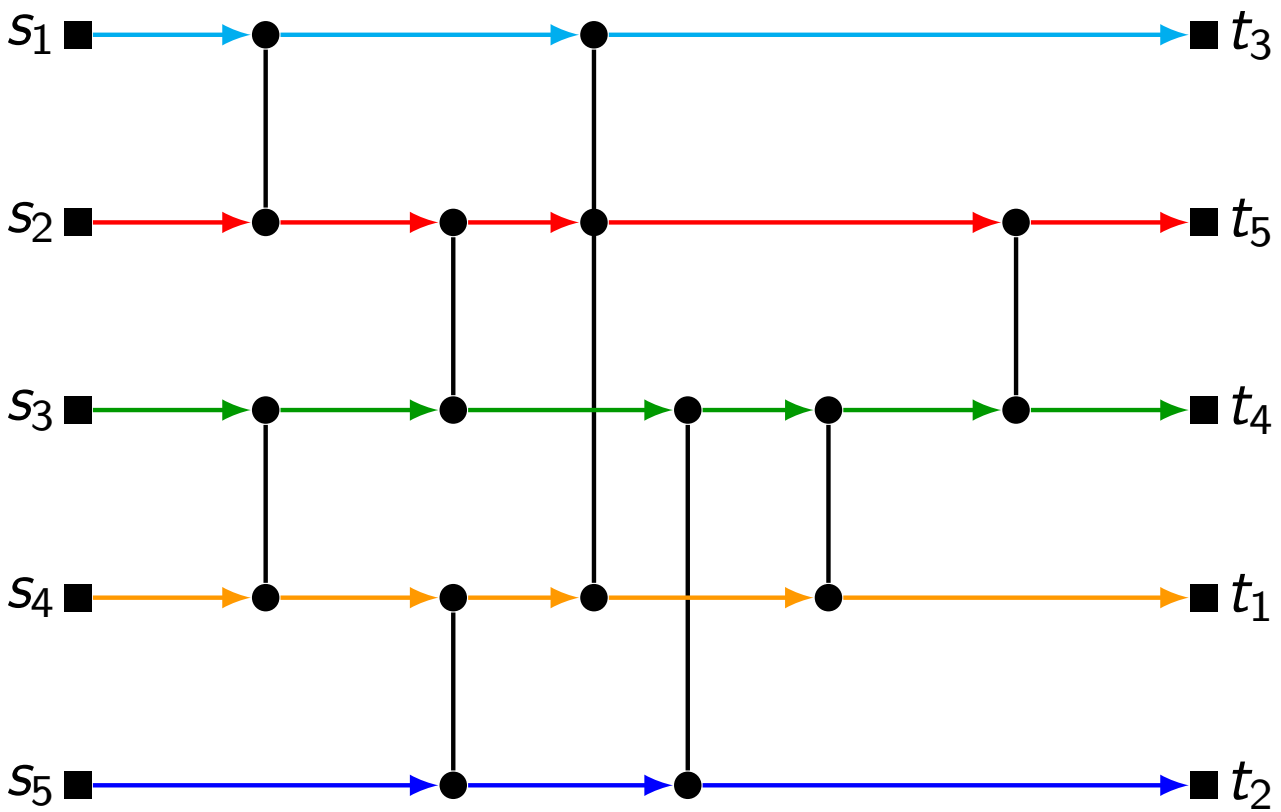

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Networks

Definition (Network)

A *network* of *width* n and *length* l is a sequence $S_1, \ldots, S_l$ of sets $S_i \subseteq [\![1, n]\!]$.

A permutation π is *valid* for $S_1, \ldots, S_l$ if:

- $l = 0$ and π is the identity,
- or $l \neq 0$ and there exists π' such that:
  - π' is valid for $S_1, \ldots, S_{l-1}$,
  - $\pi'(i) \notin S_l \implies \pi(i) = \pi'(i)$.

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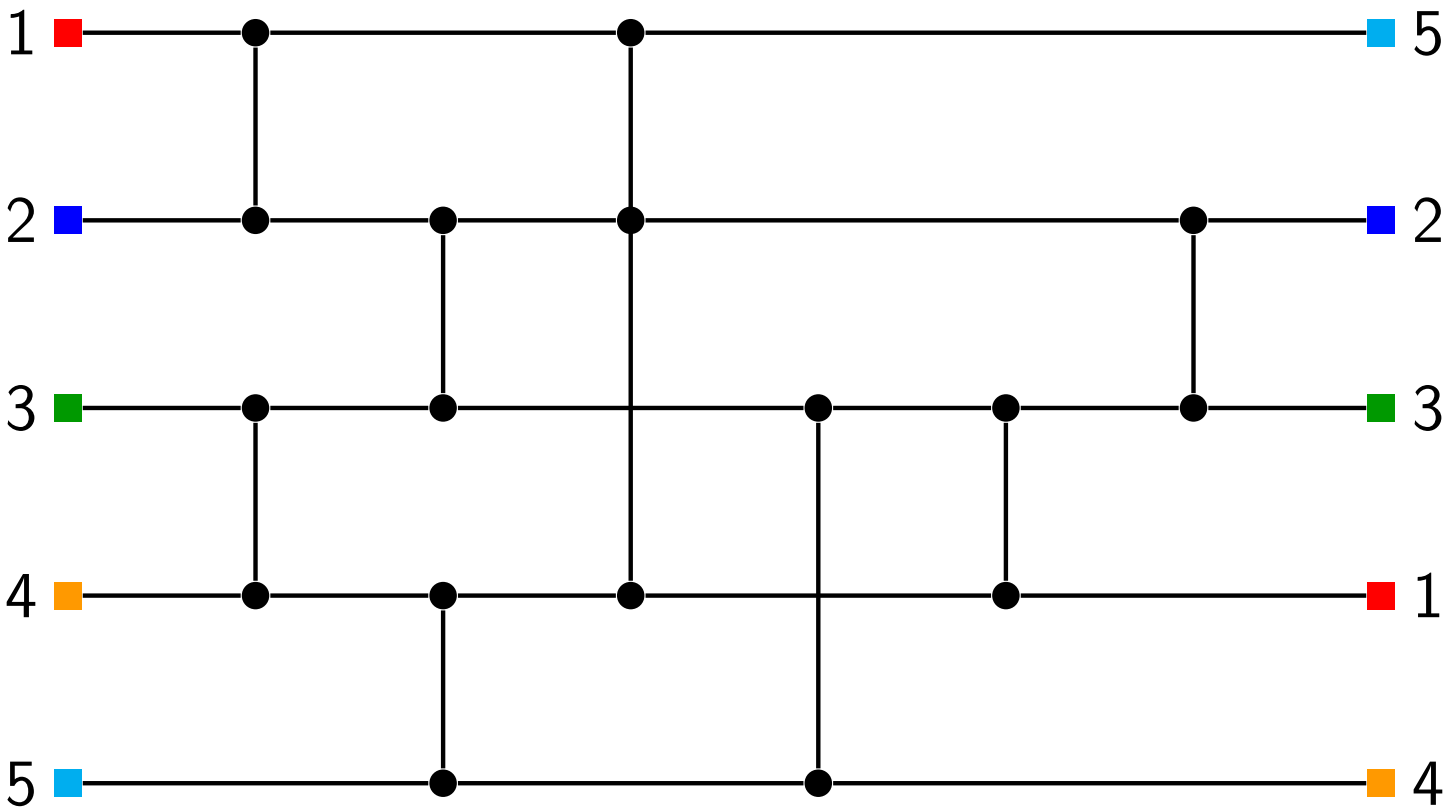
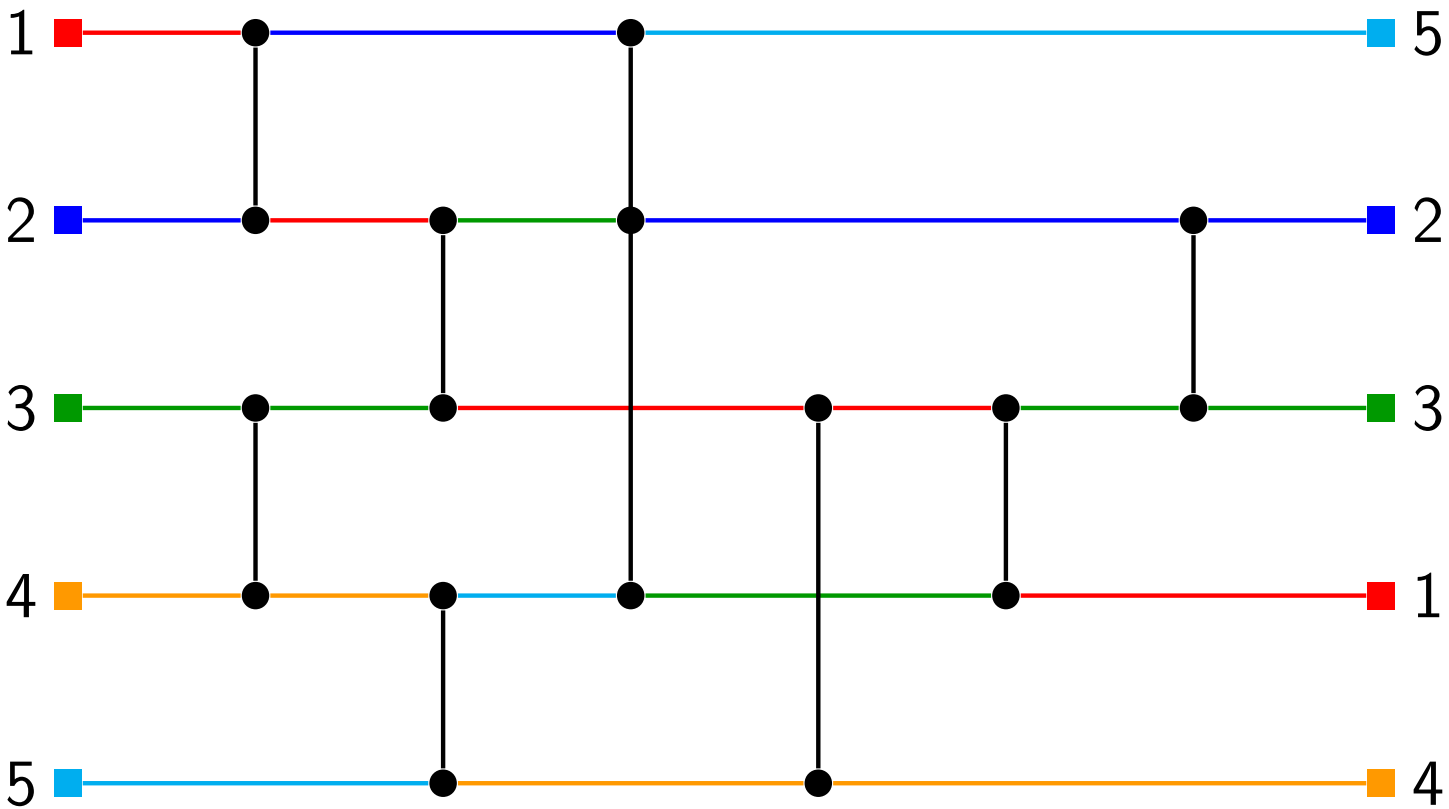
Examples of validity



Examples of validity



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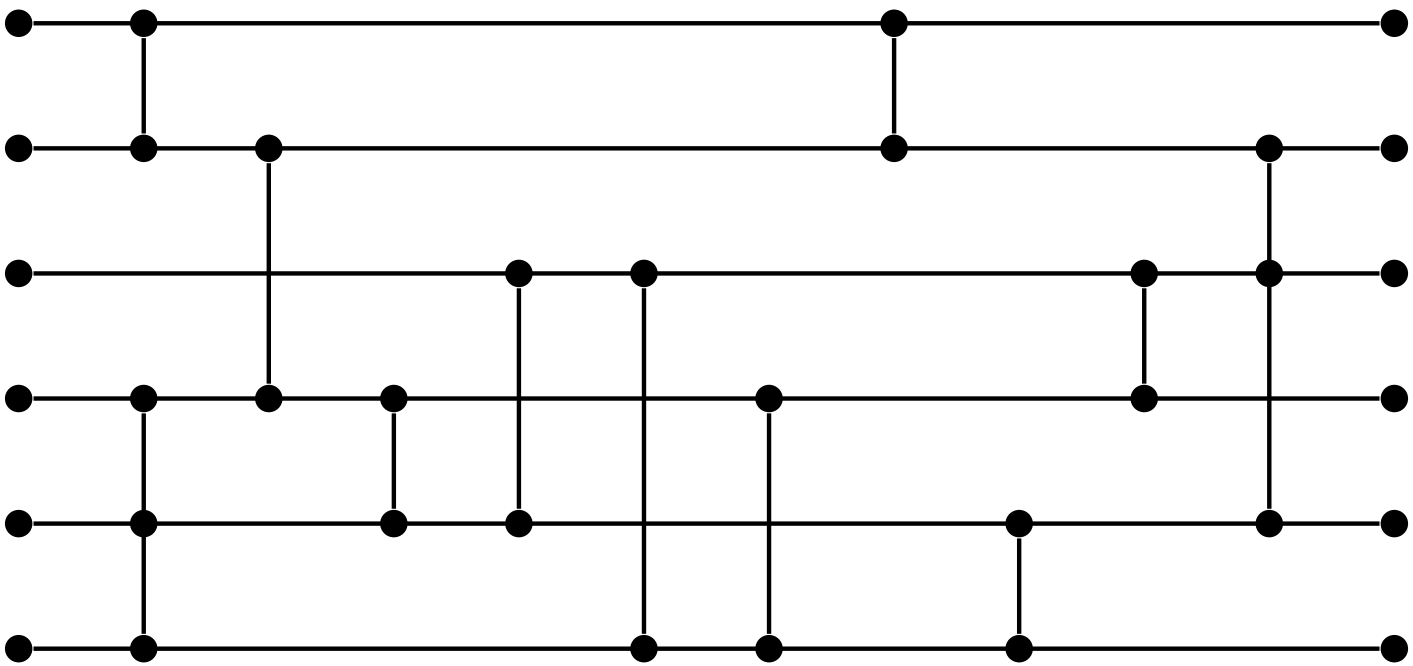
Subnetworks

Definition

Let be $i < j \in [\![1, l]\!]$, $A \subseteq [\![1, n]\!]$ and $f : A \to [\![1, |A|]\!]$ injective. Suppose that for all $k \in [\![i, j]\!]$, $A \cap S_k = \emptyset$ or $S_k \subseteq A$. Then $f(S_i \cap A), \ldots, f(S_j \cap A)$ is a *subnetwork* of $S_1, \ldots, S_n$.



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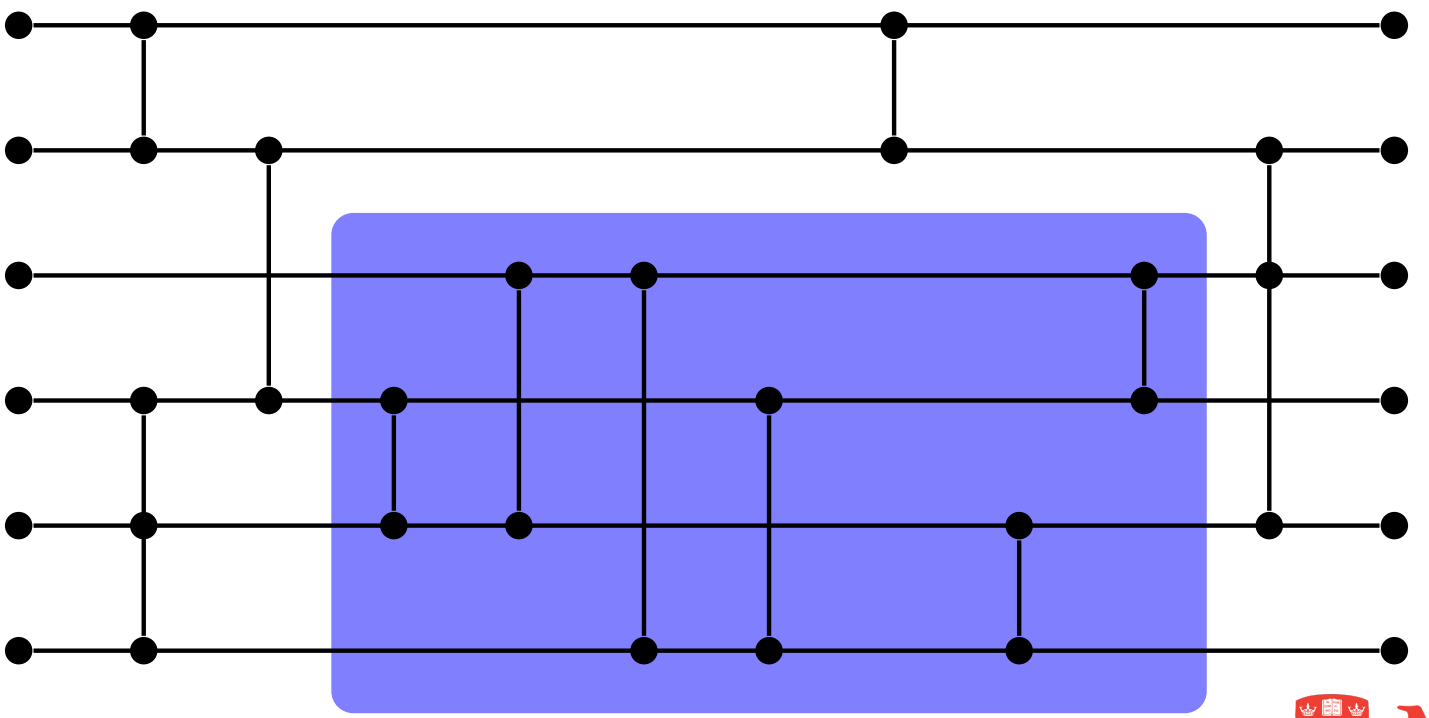

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Universal networks

Definition

A network of width n is *universal* if every permutation of σ_n is valid for this network.

Theorem

For every n, there is a constant M_n such that every network of width n and length M_n contains a non-trivial universal subnetwork.

Consequence: for a given k, only a finite number of graphs to check.

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Computations

Computation of all the minimal planar networks of order $k \in \{4, 5, 6\}$:

- 3 non-trivial graphs for $k = 4$.
- ≈ 2000 for $k = 5$.
- $\approx 40M$ for $k = 6$

Three weeks of computation on an eight-core machine, to find...

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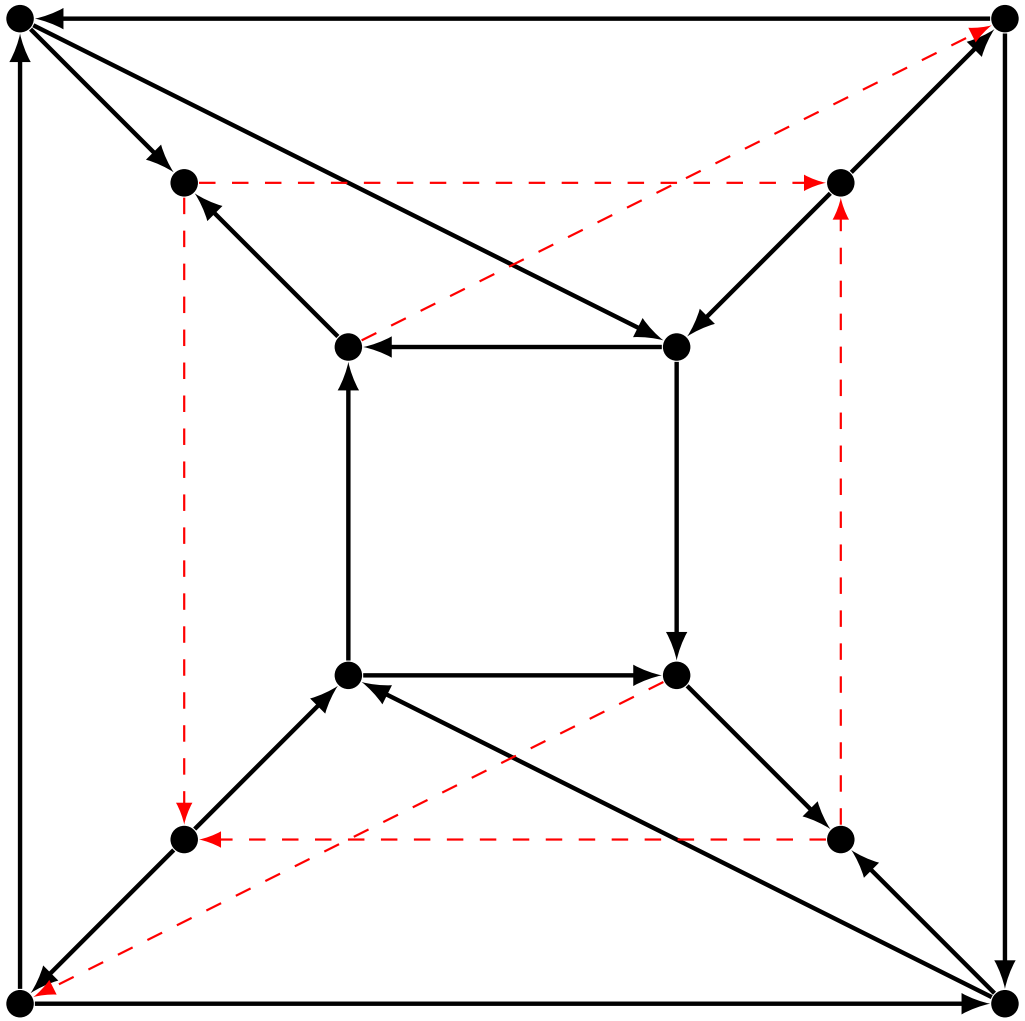
A counter-example



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A polynomial case

Theorem (N, 2009)

If G is planar and acyclic, $|H|$ is fixed, and $G + H$ is Eulerian, Arc-Disjoint-Paths *is solvable in time $O(f(R)^{k^2} k^3 n)$, where f is a polynomial function, $R = max_{h \in H} r(h)$.*

Proof :

- Uncross the paths,
- Study the intersections of the paths,
- Locally route every vertex.

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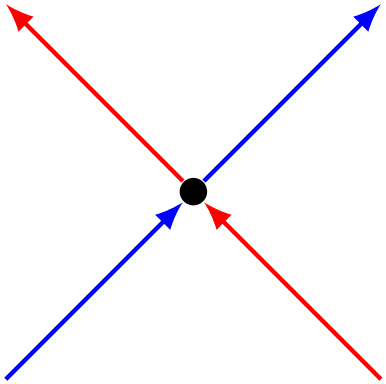
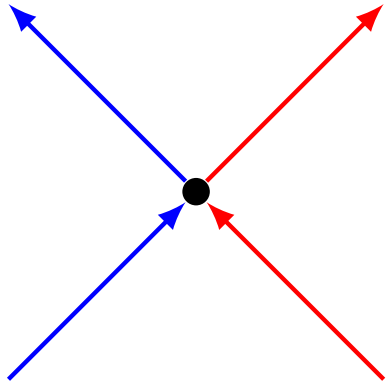
Uncrossing the paths

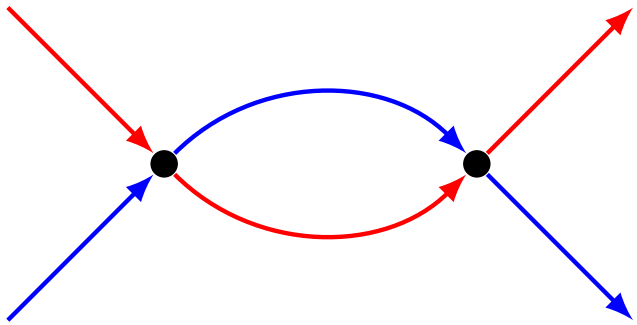


Crossing.

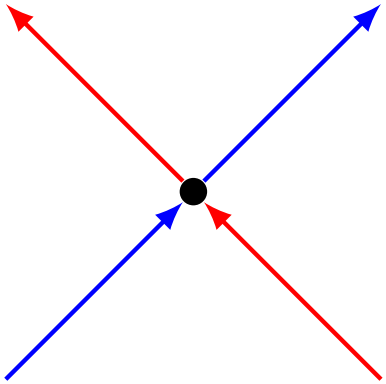


Not crossing.

Uncrossing : two paths are crossed at most once, in their first common vertex.



Uncrossing the paths



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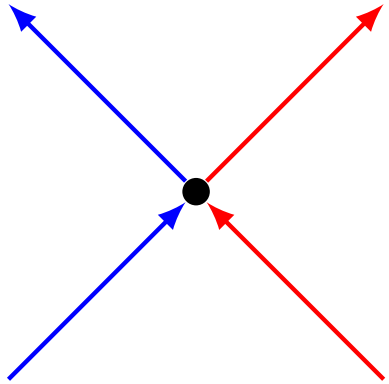

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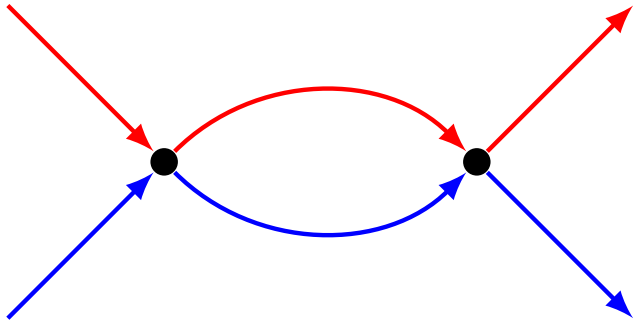

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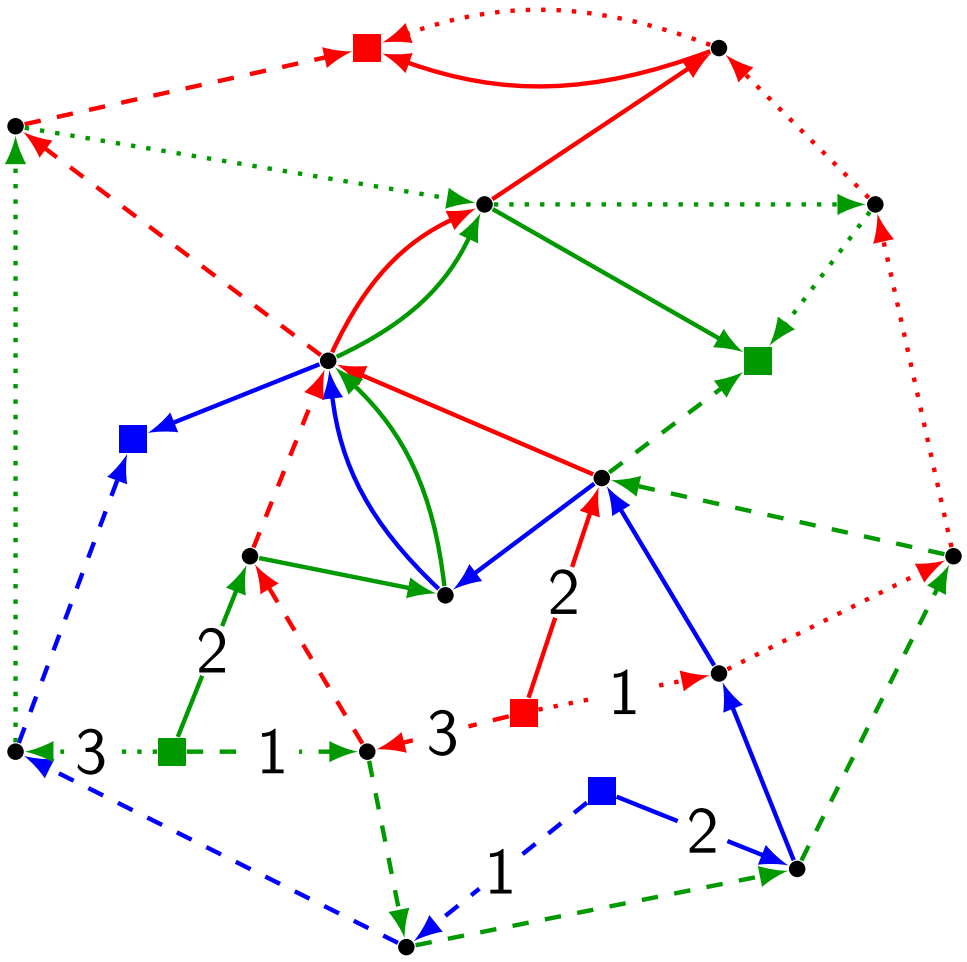

	R1	R2	R3
B1	-	-	-
B2	C	C	C

	R1	R2	R3
G1	C	LR	RR
G2	-	C	C
G3	RR	C	C

	G1	G2	G3
B1	C	-	C
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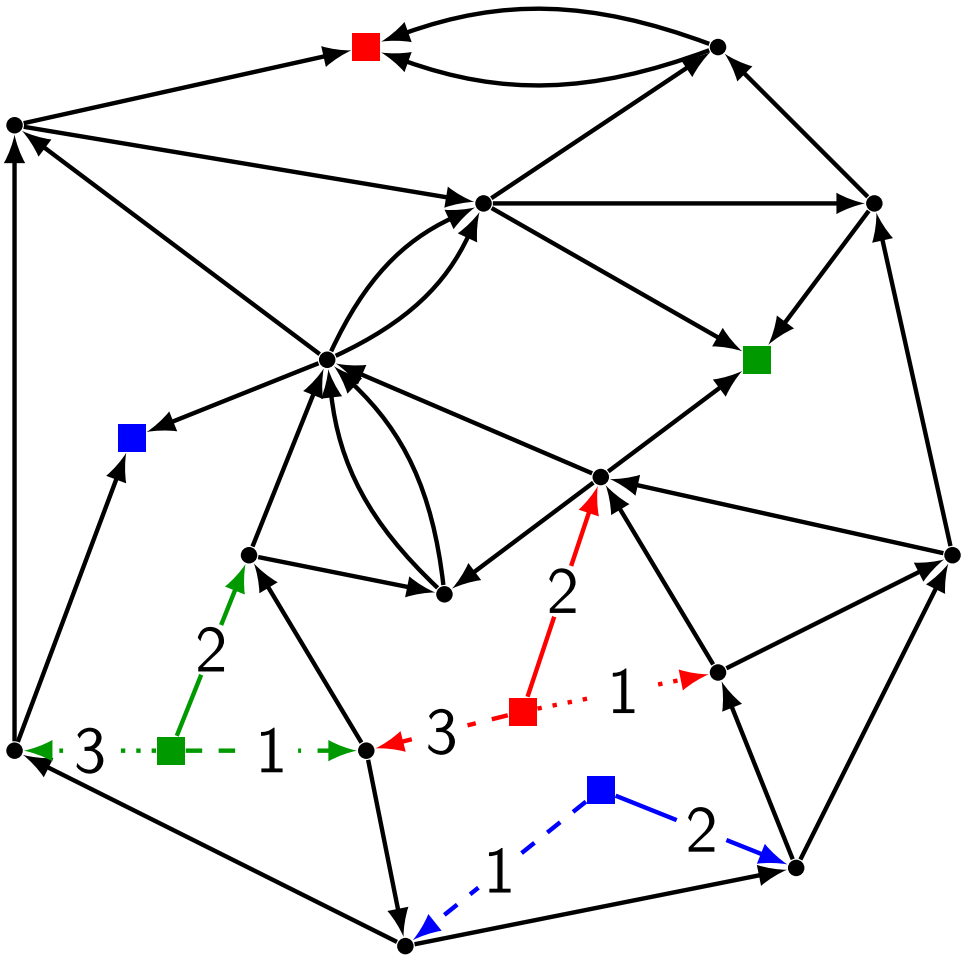

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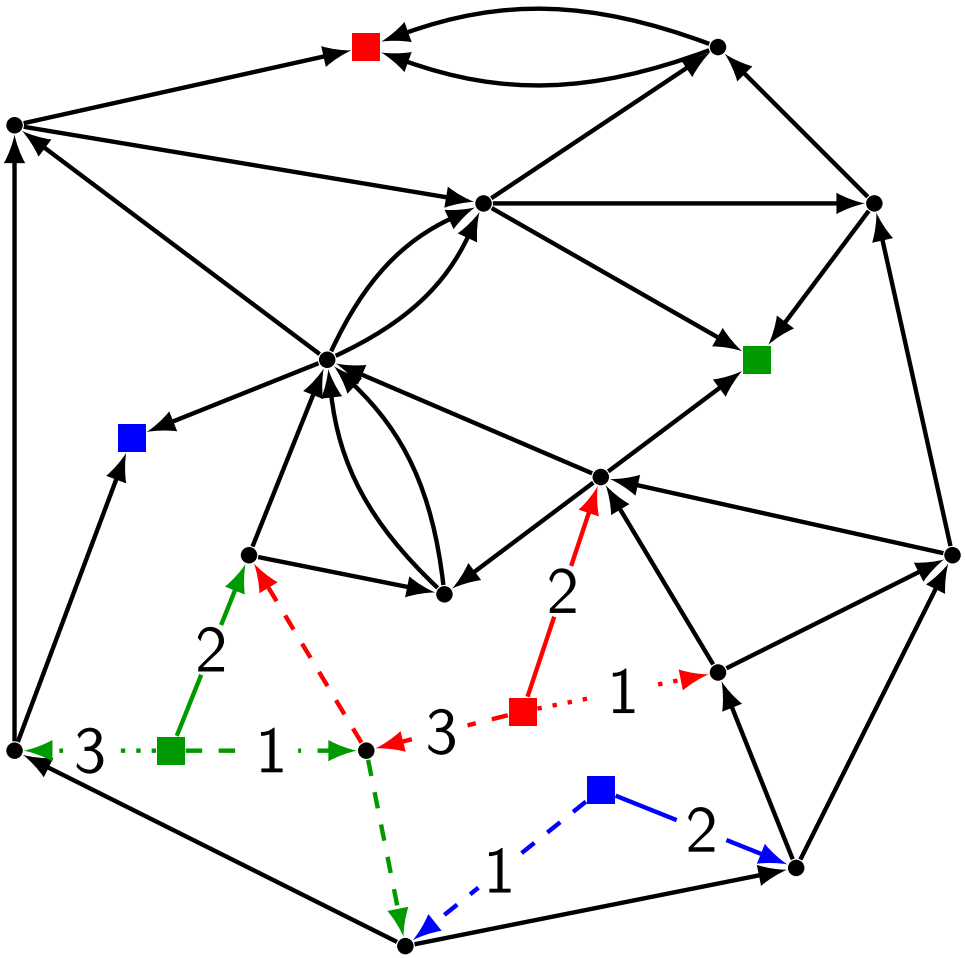

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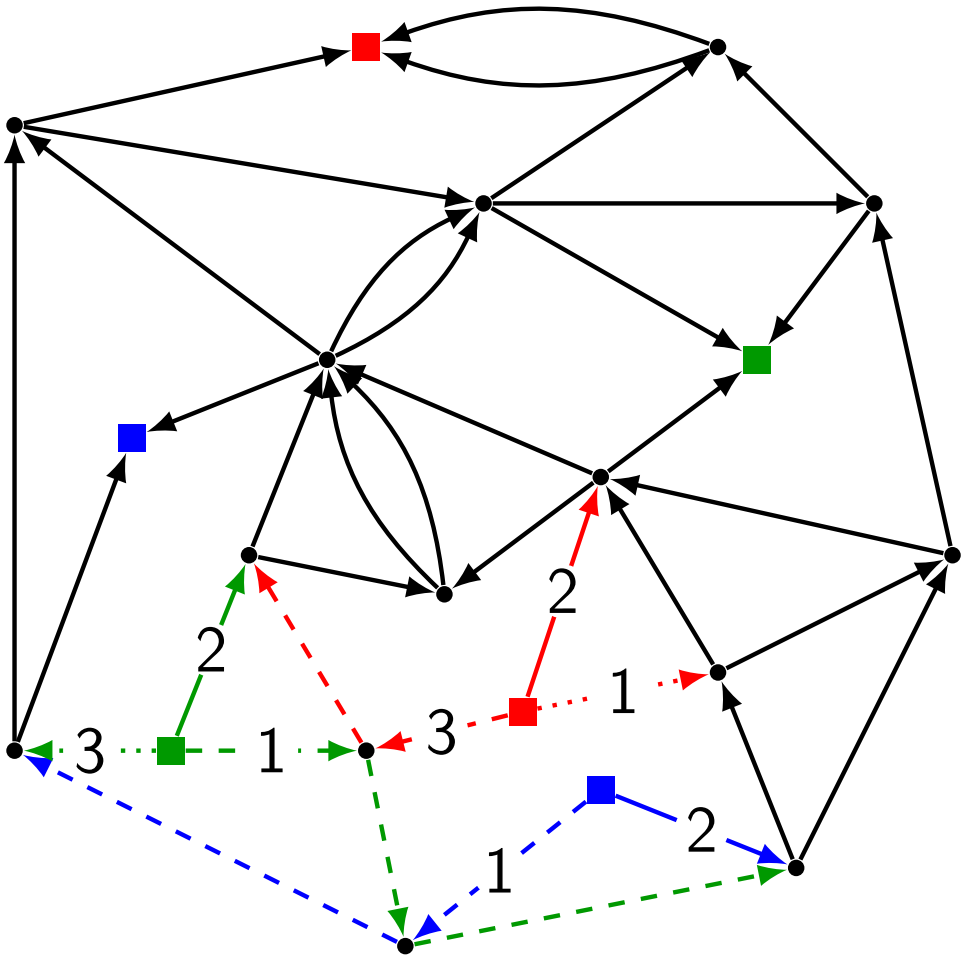

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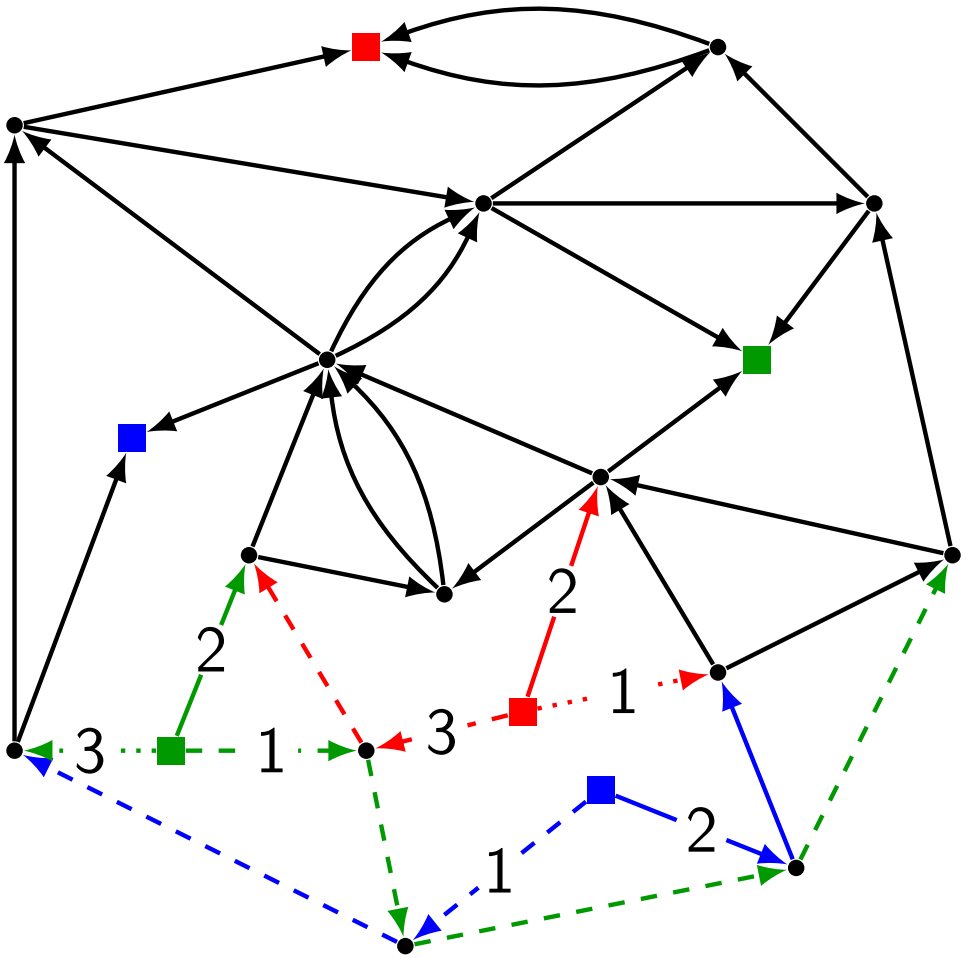

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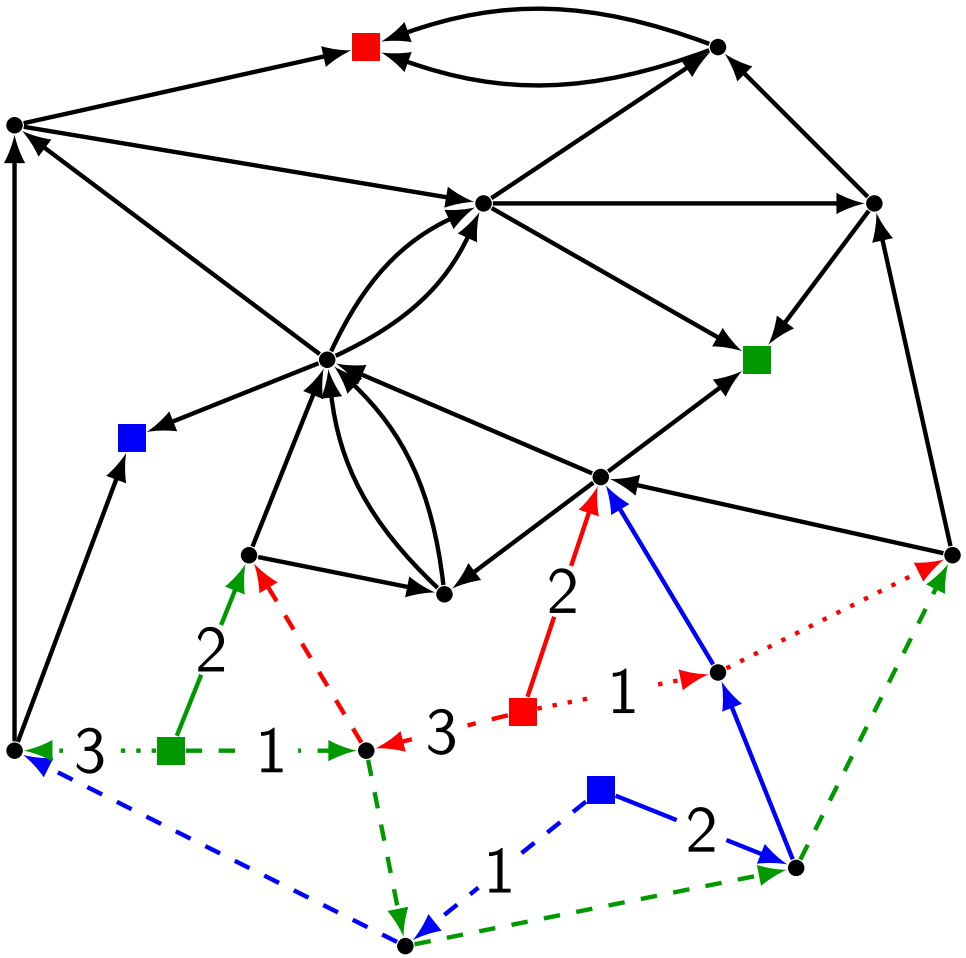

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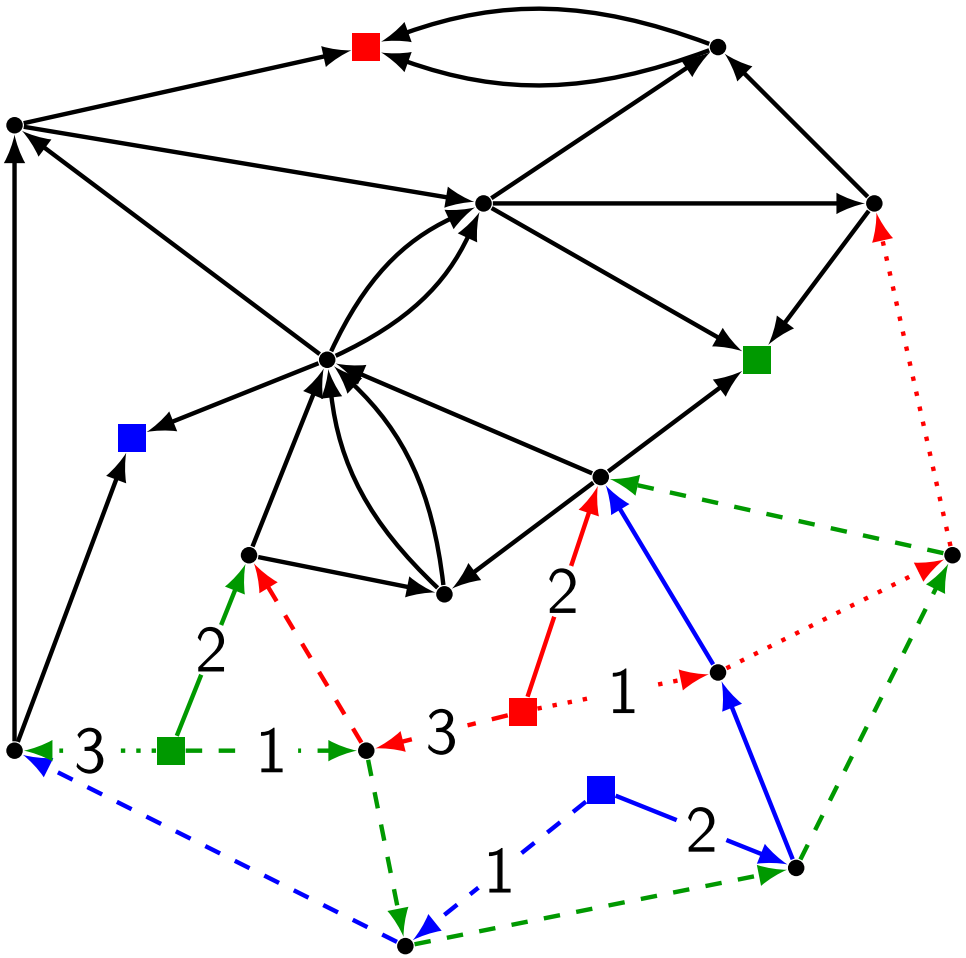

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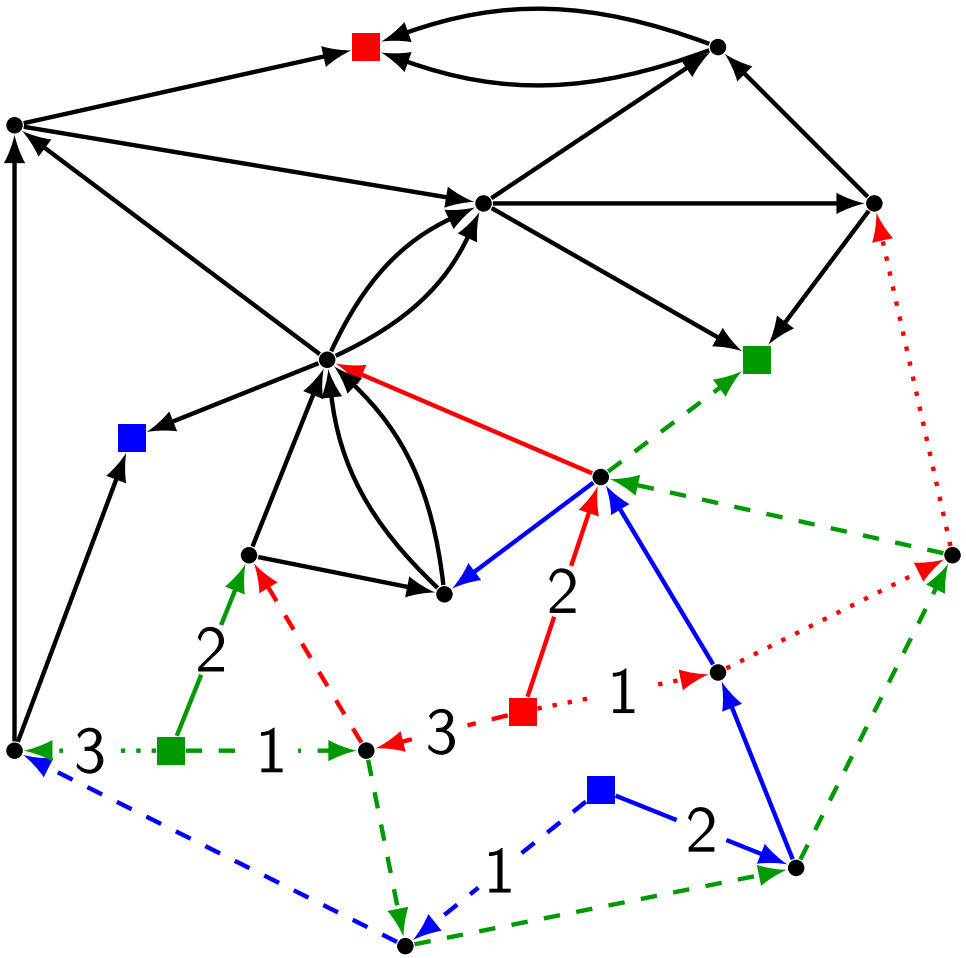

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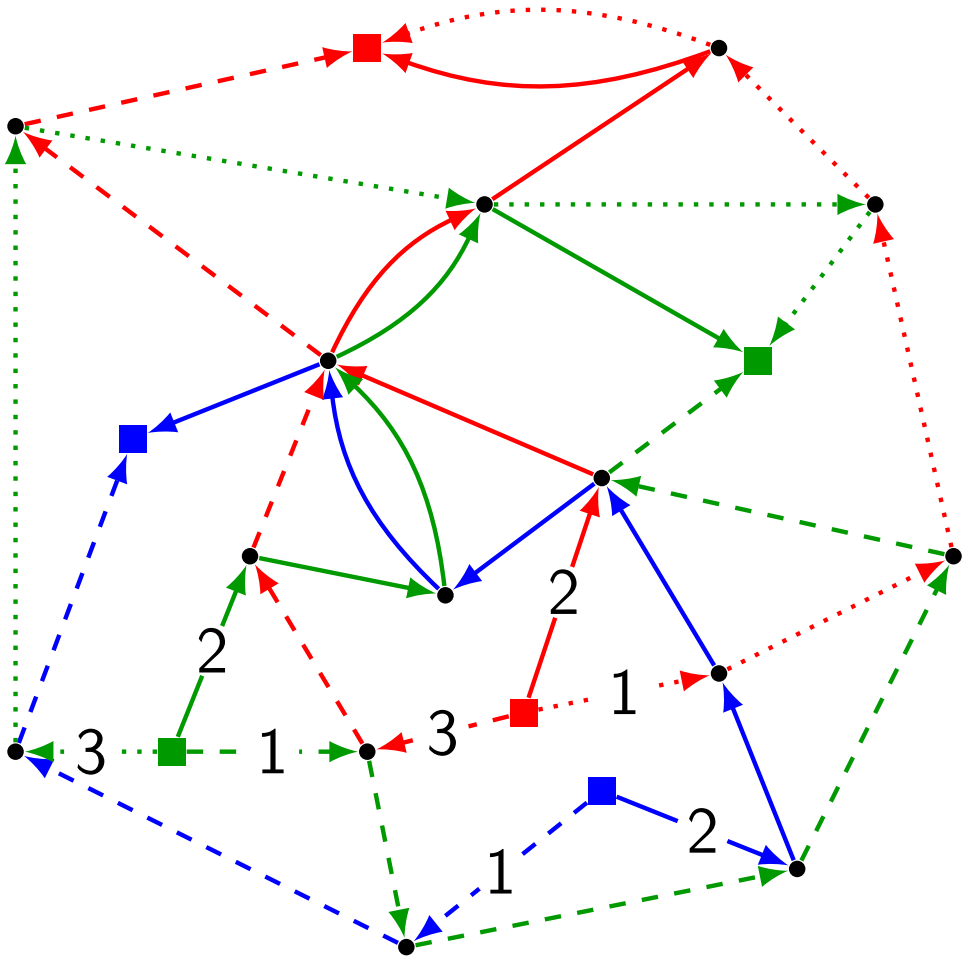

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	G1	G2	G3
B1	C	-	C
B2	RL	C	-

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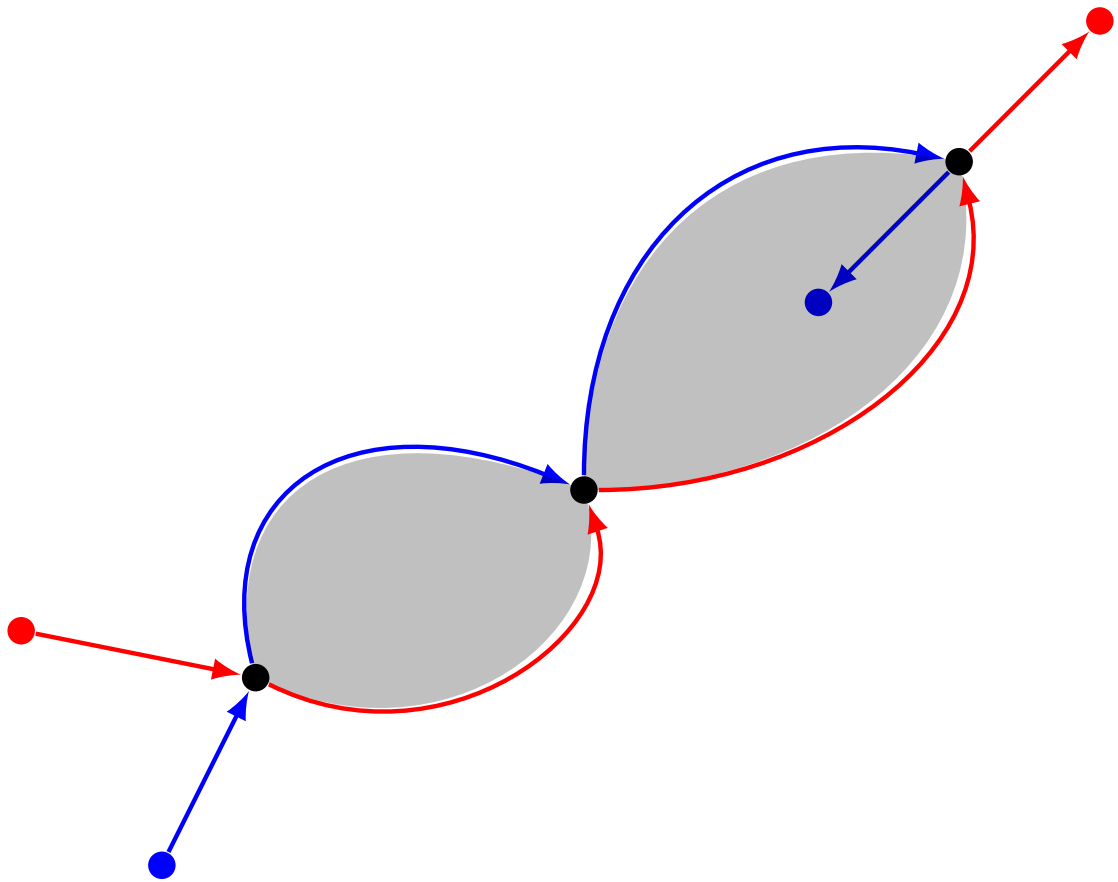
Second intersection

What happens at the second (or more) intersection of two paths ?



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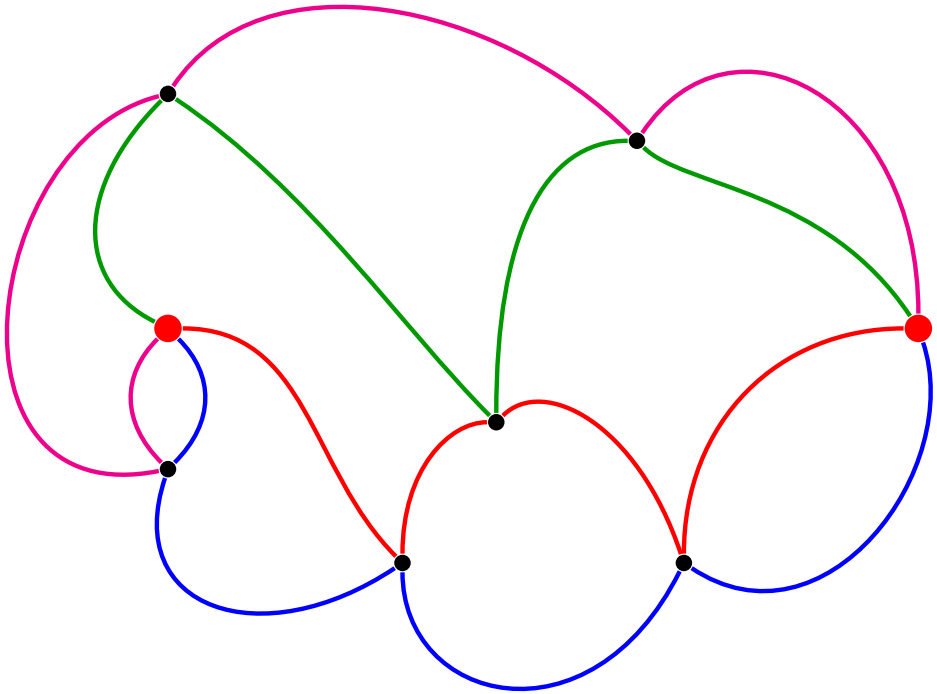
How do the paths cross ?



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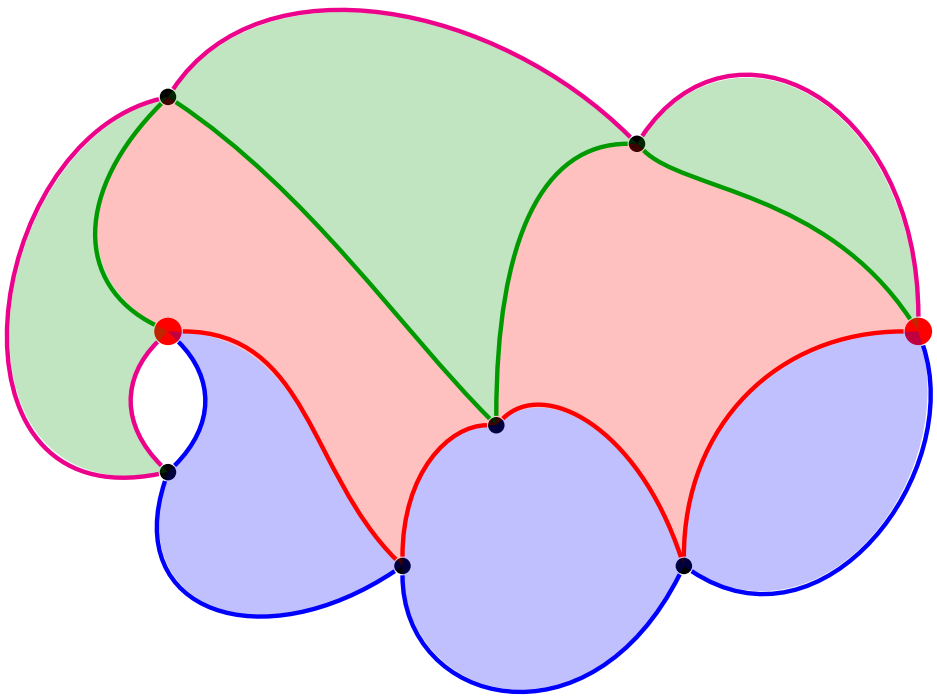

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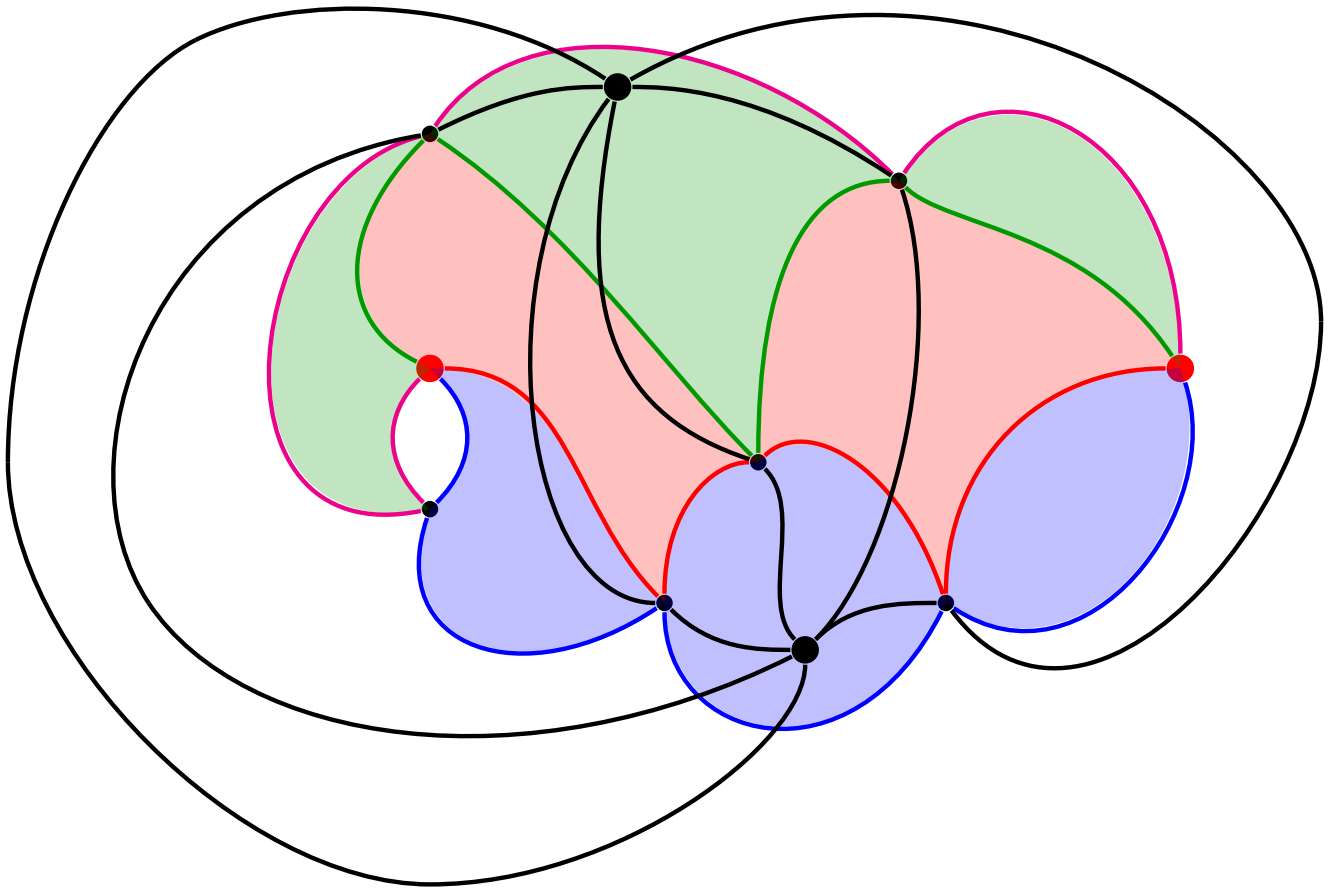

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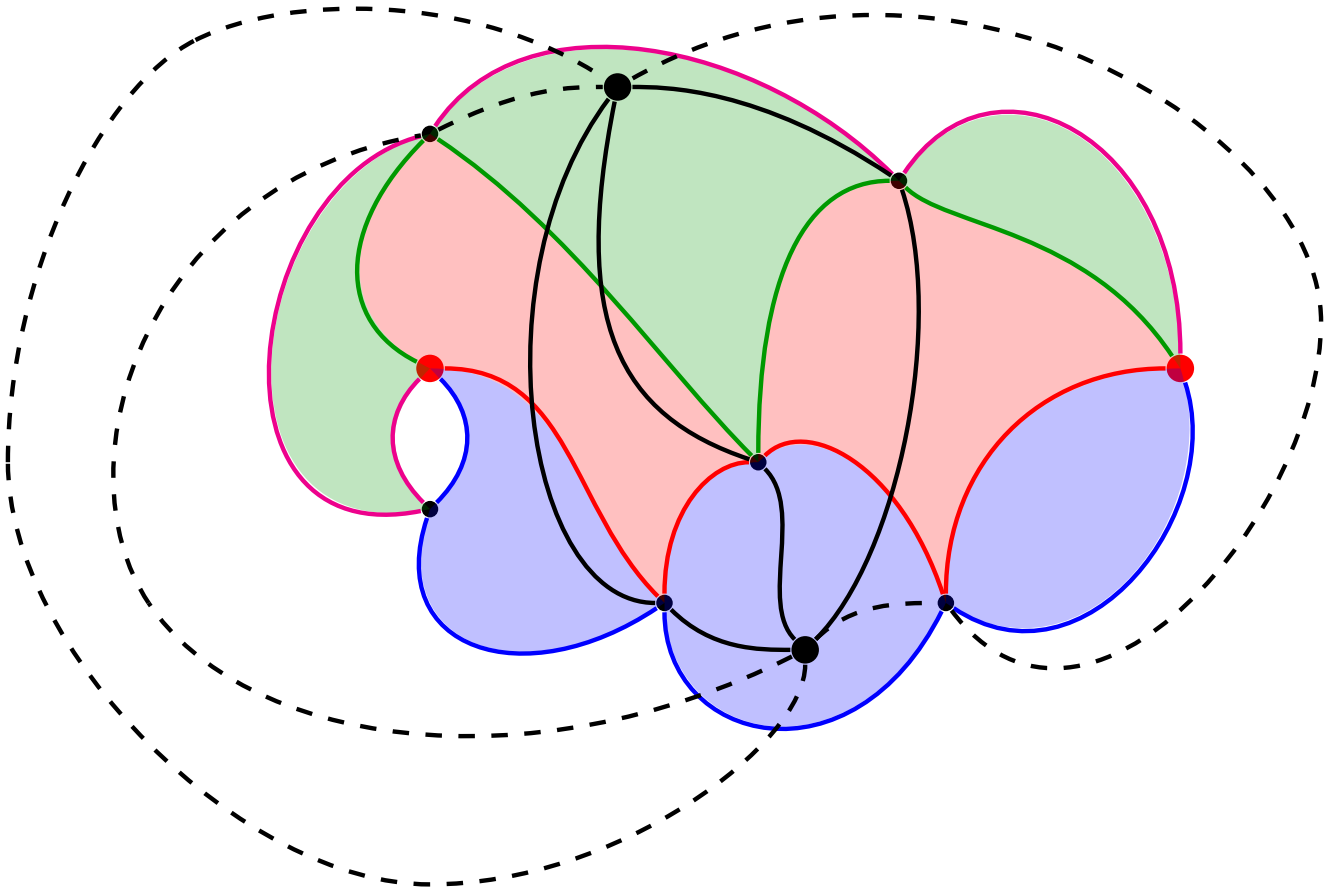

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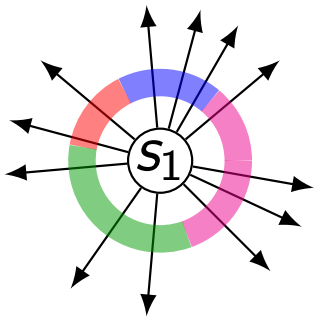
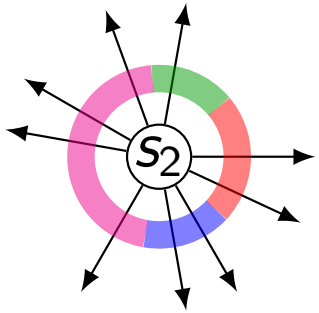
For a pair of demands





	A_2	B_2	C_2	D_2
A_1	C	C	L/L	L/R
B_1	C	C	R/L	R/R
C_1	L/L	L/R	C	C
D_1	R/L	R/R	C	C

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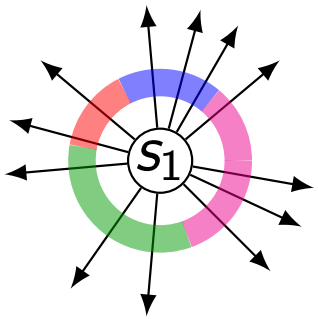
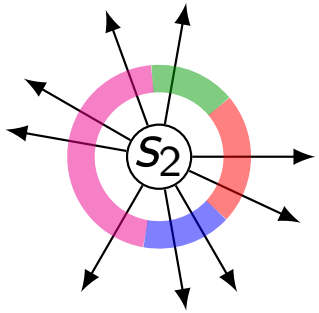
For a pair of demands

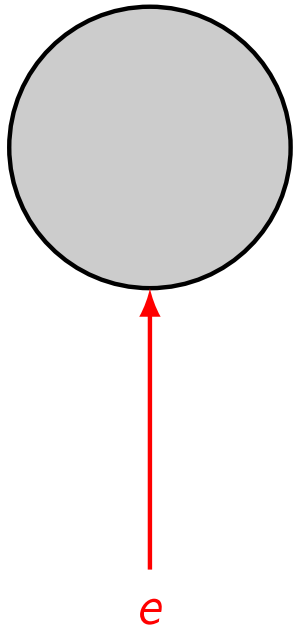




	A_2	B_2	C_2	D_2
A_1	C	C	L/L	L/R
B_1	C	C	R/L	R/R
C_1	L/L	L/R	C	C
D_1	R/L	R/R	C	C

For each pair of demand arcs h_1, h_2,
at most $r(h_1)^4 r(h_2)^4$ possibilities

How to route a vertex?



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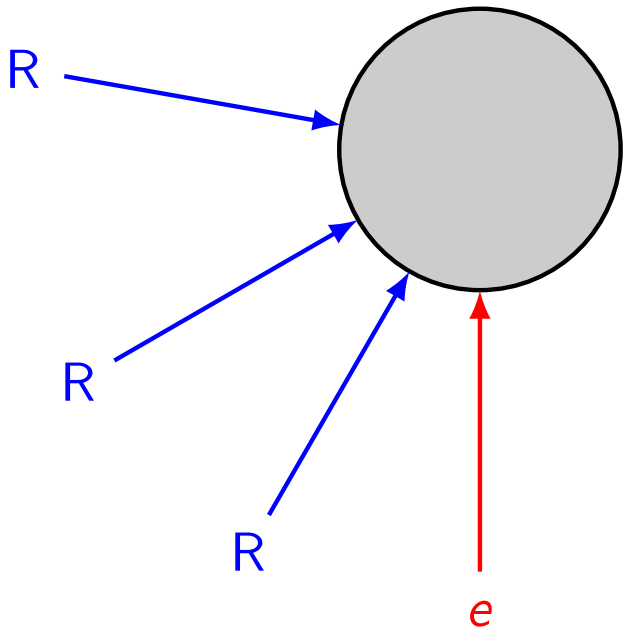
How to route a vertex?



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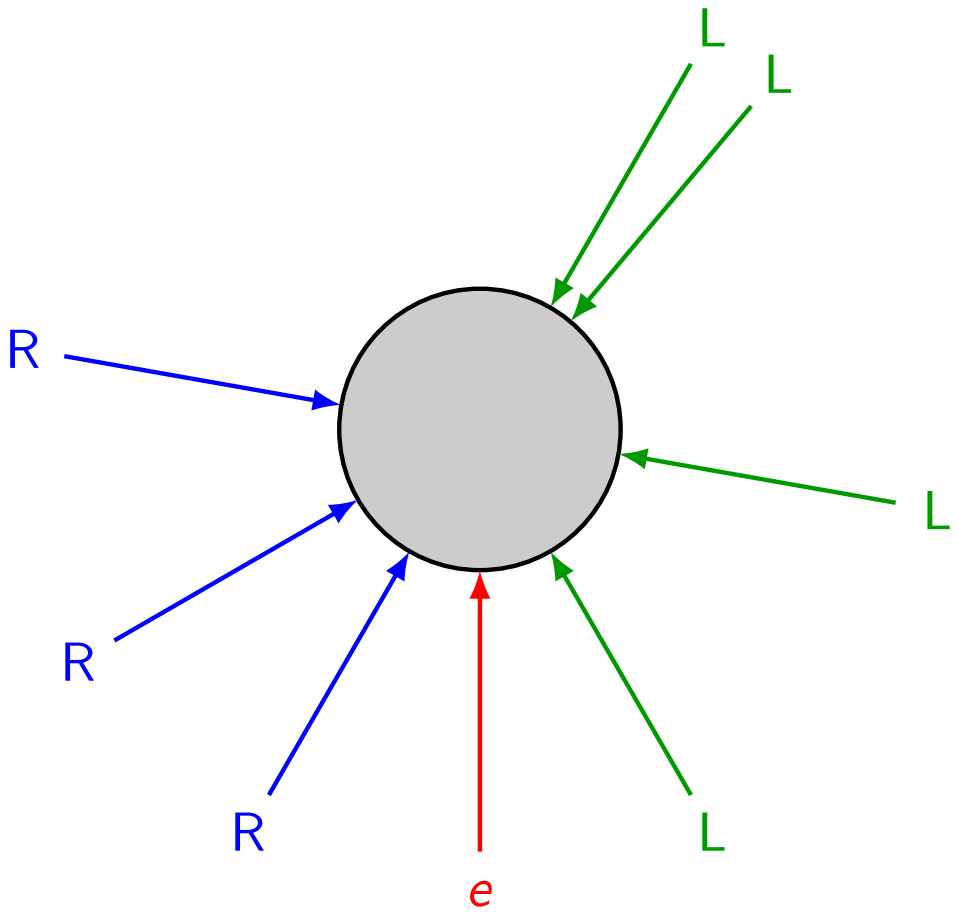

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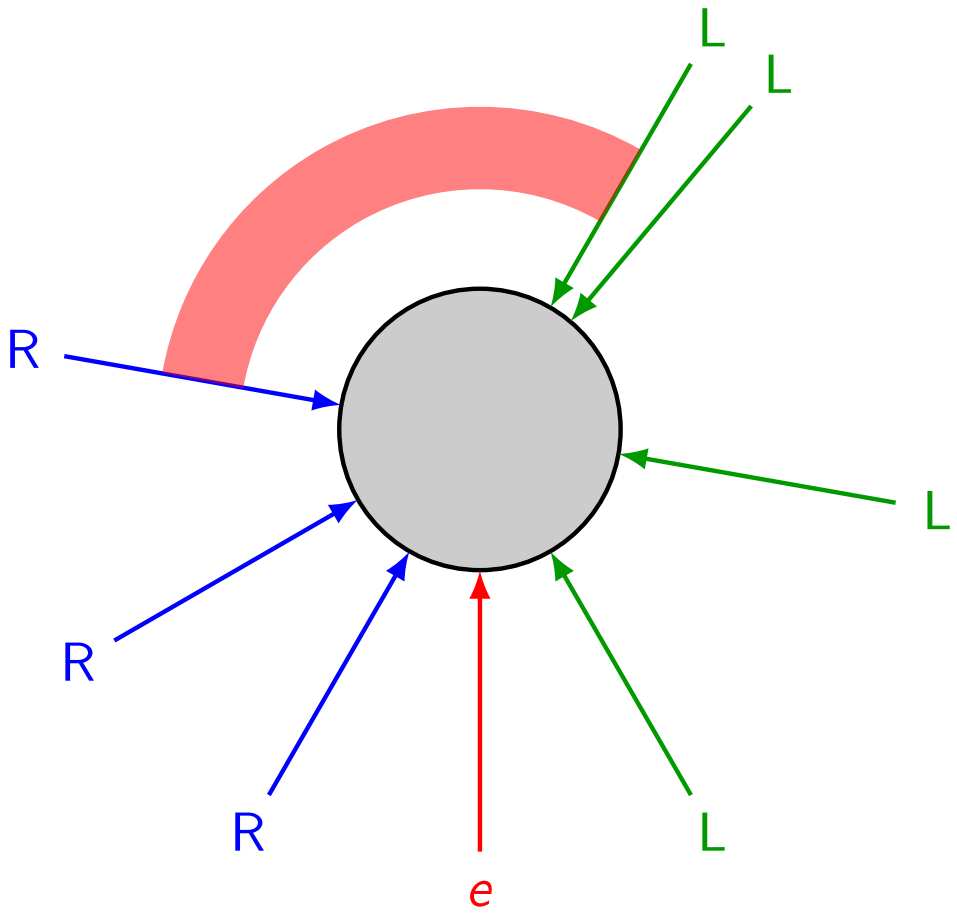
How to route a vertex?



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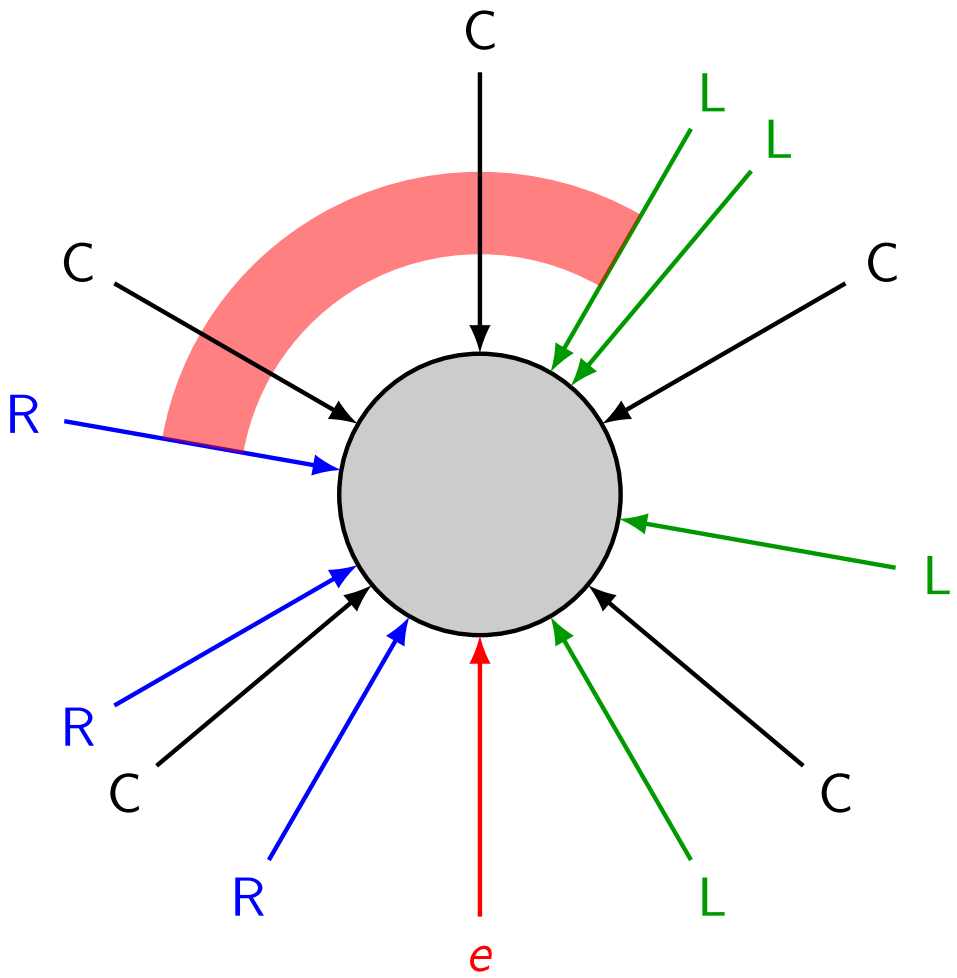

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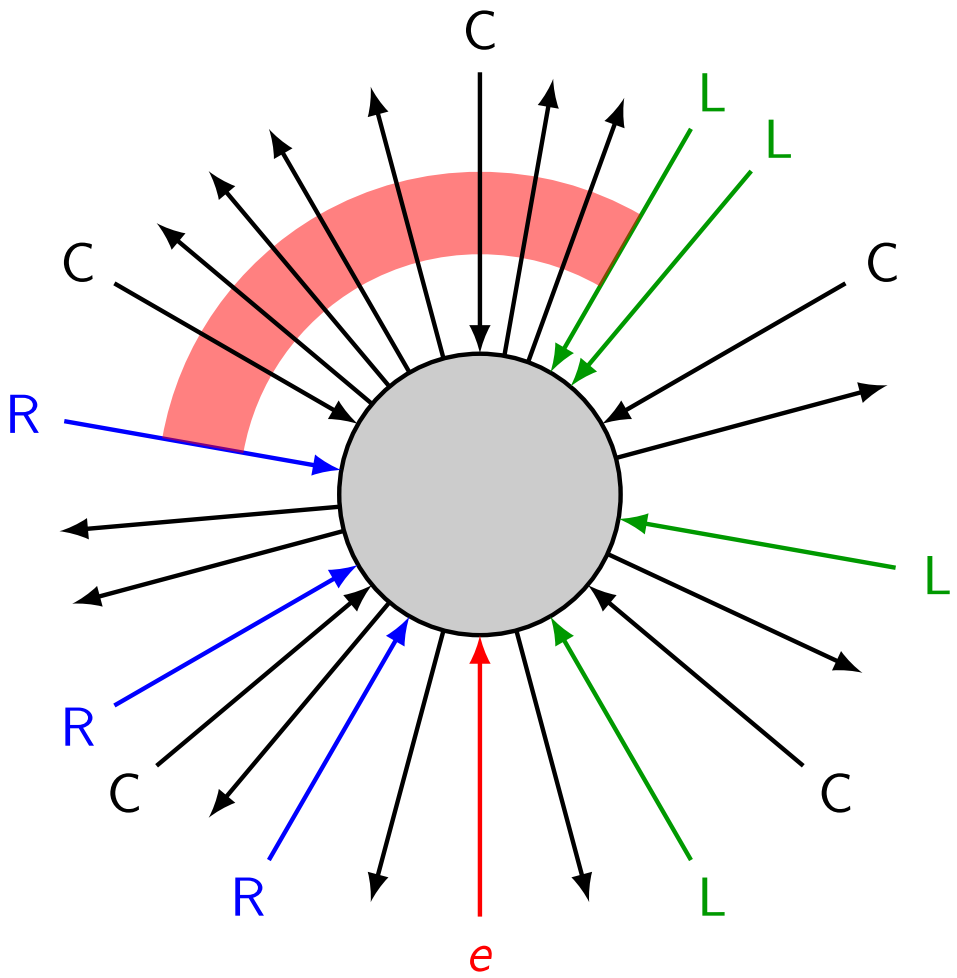

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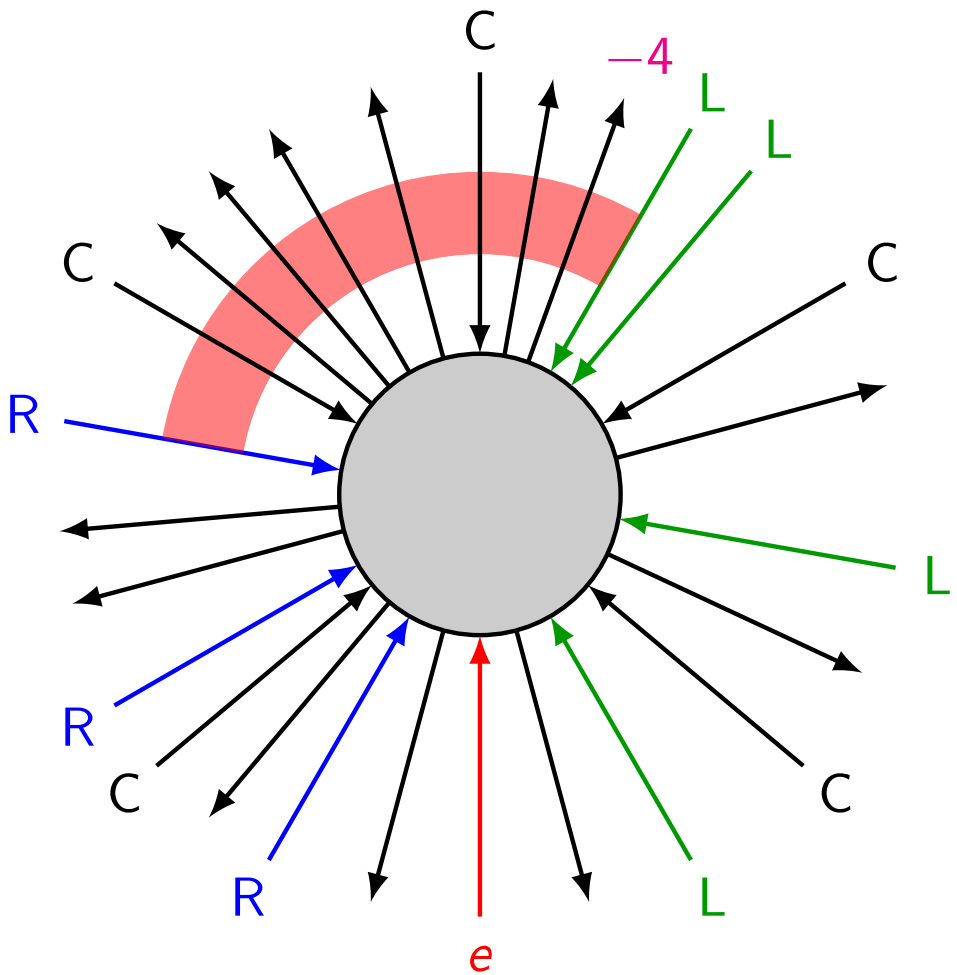

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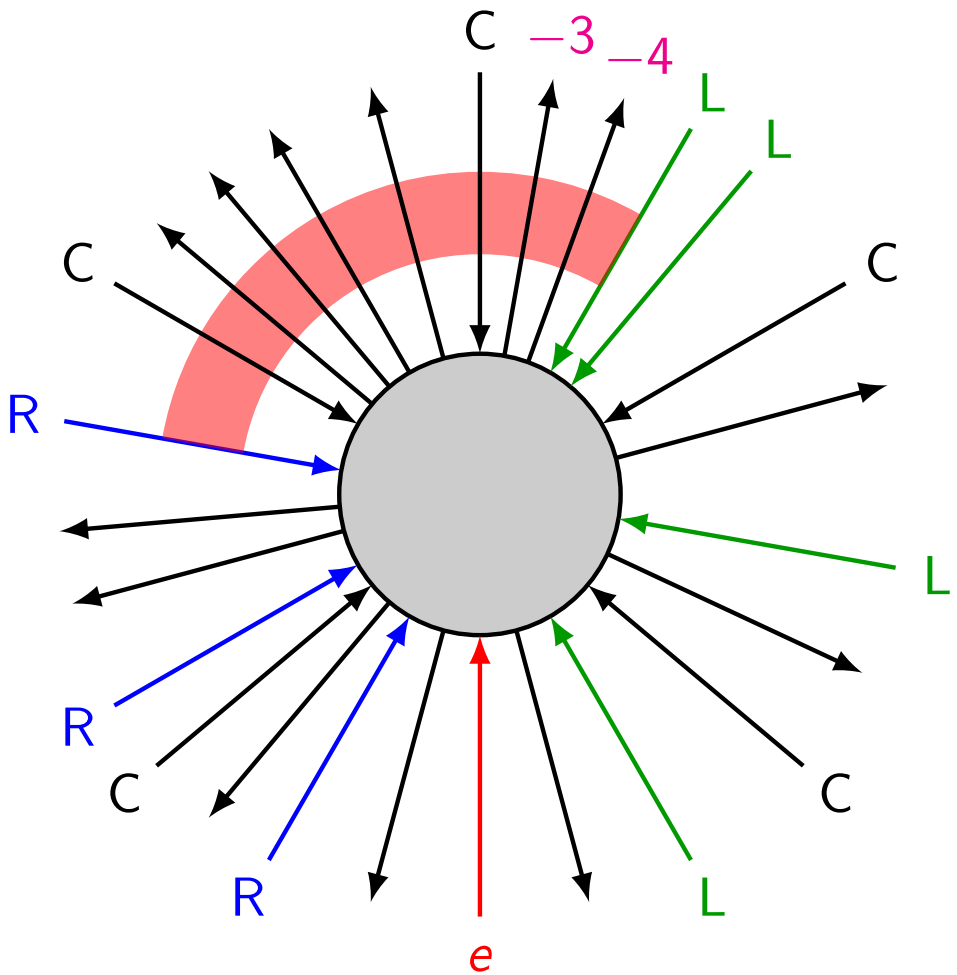

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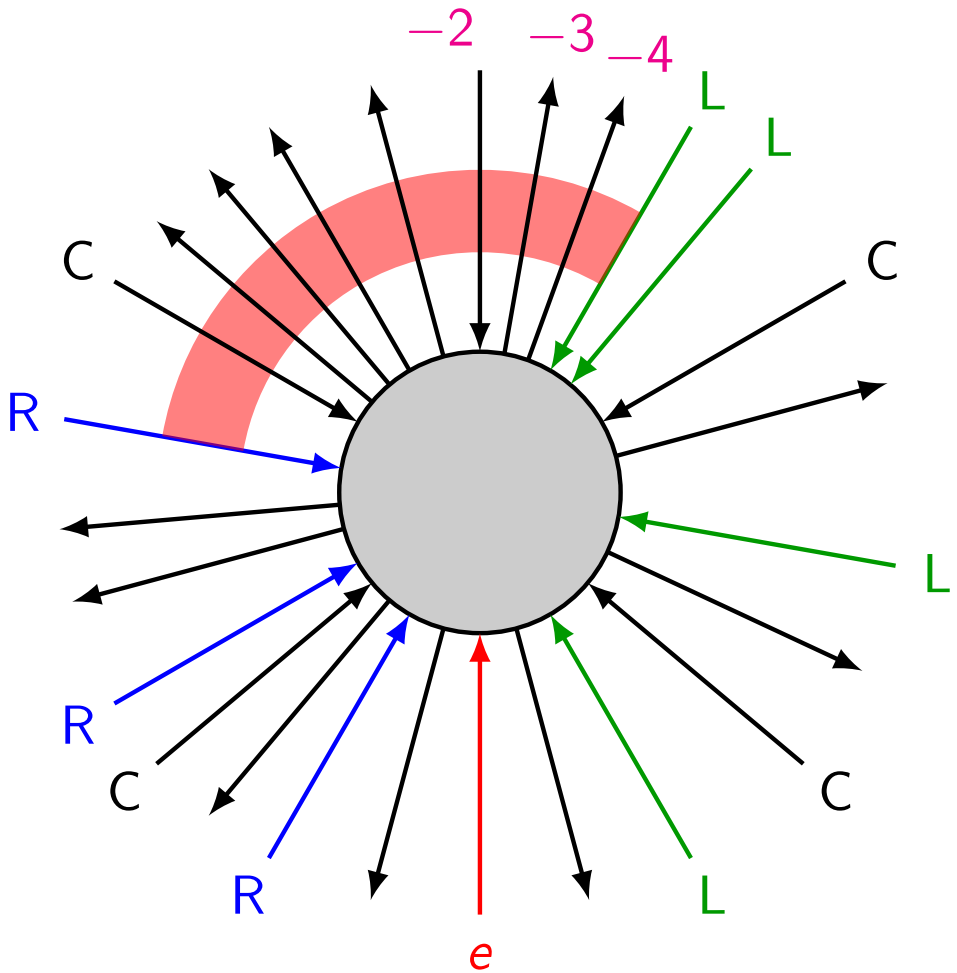

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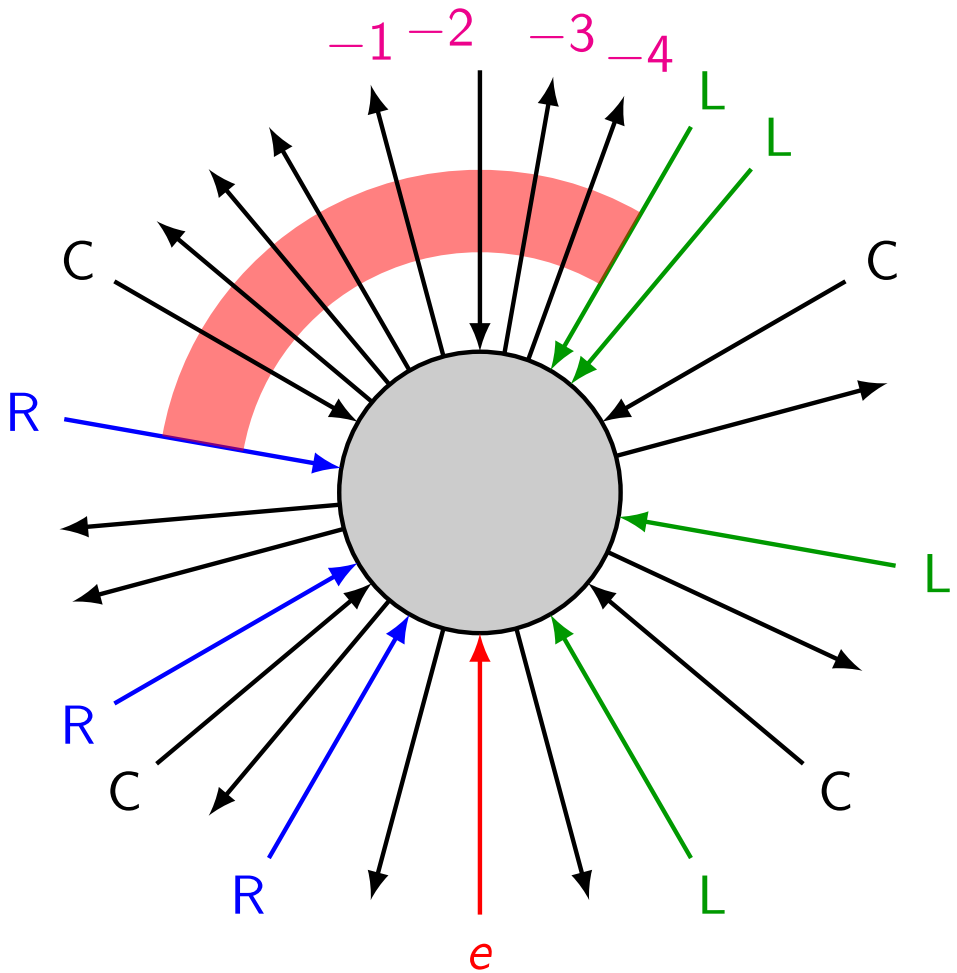

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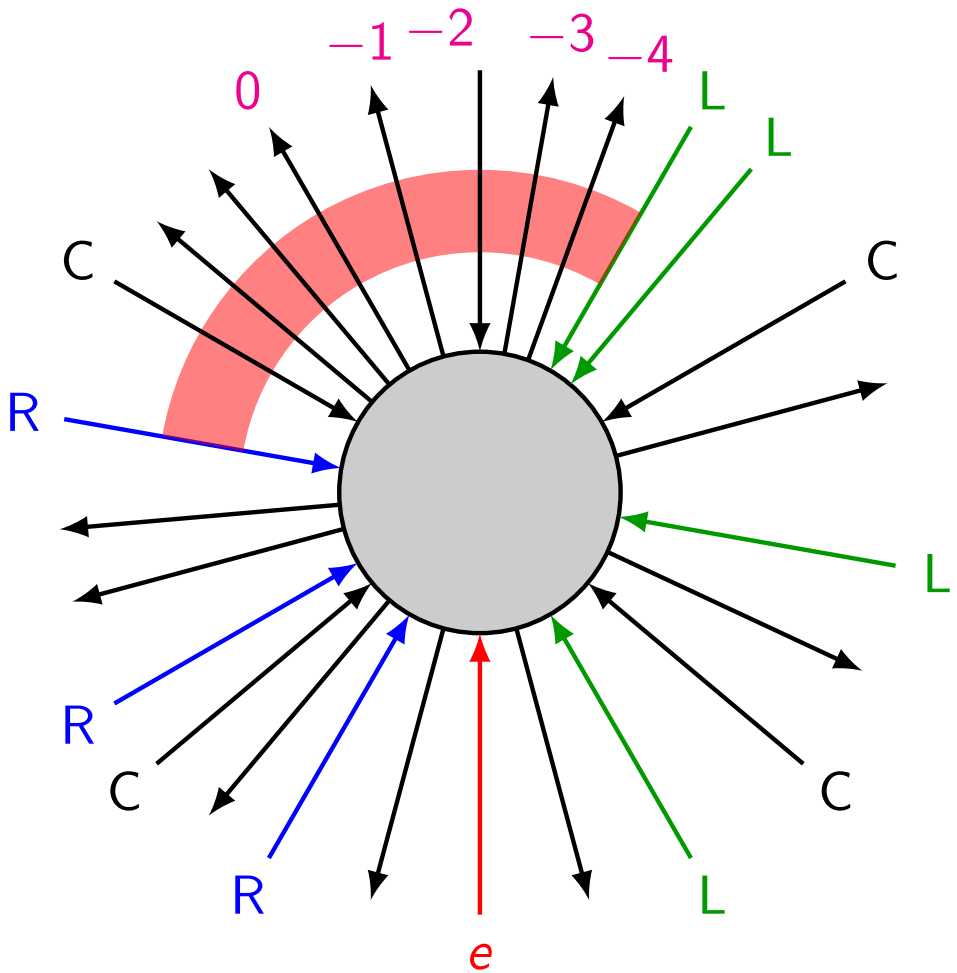

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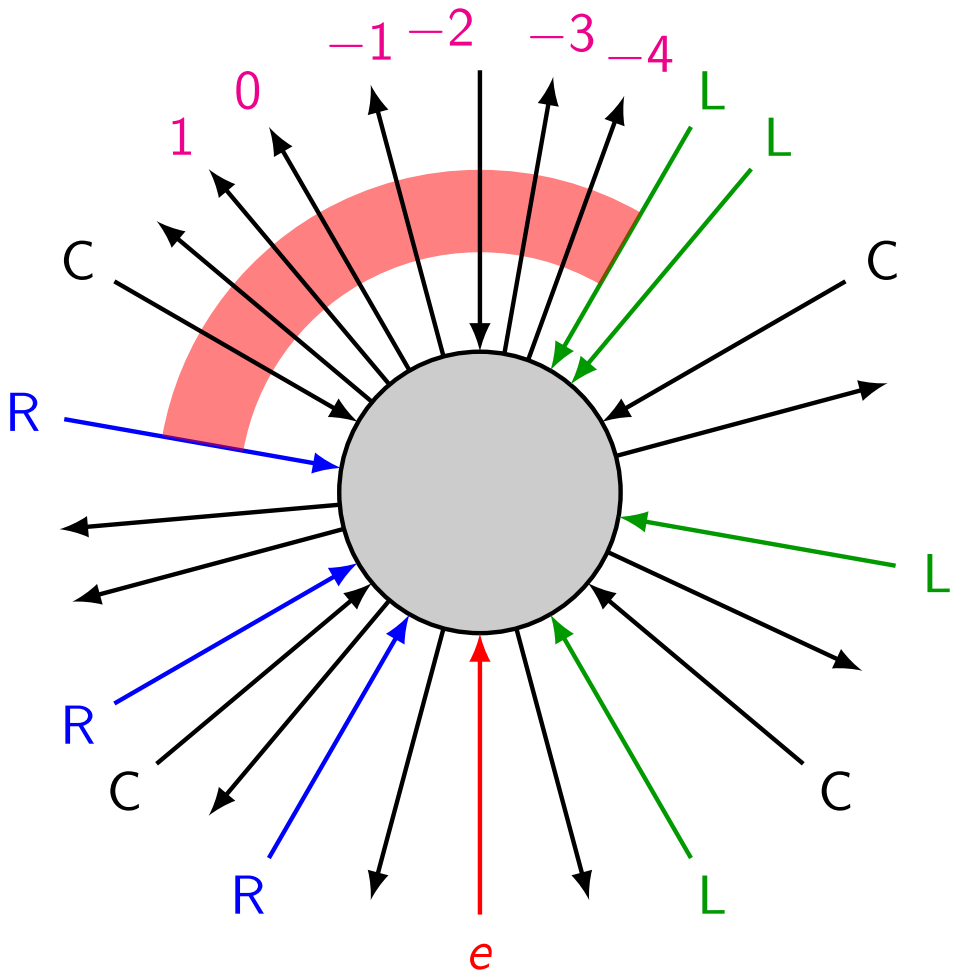

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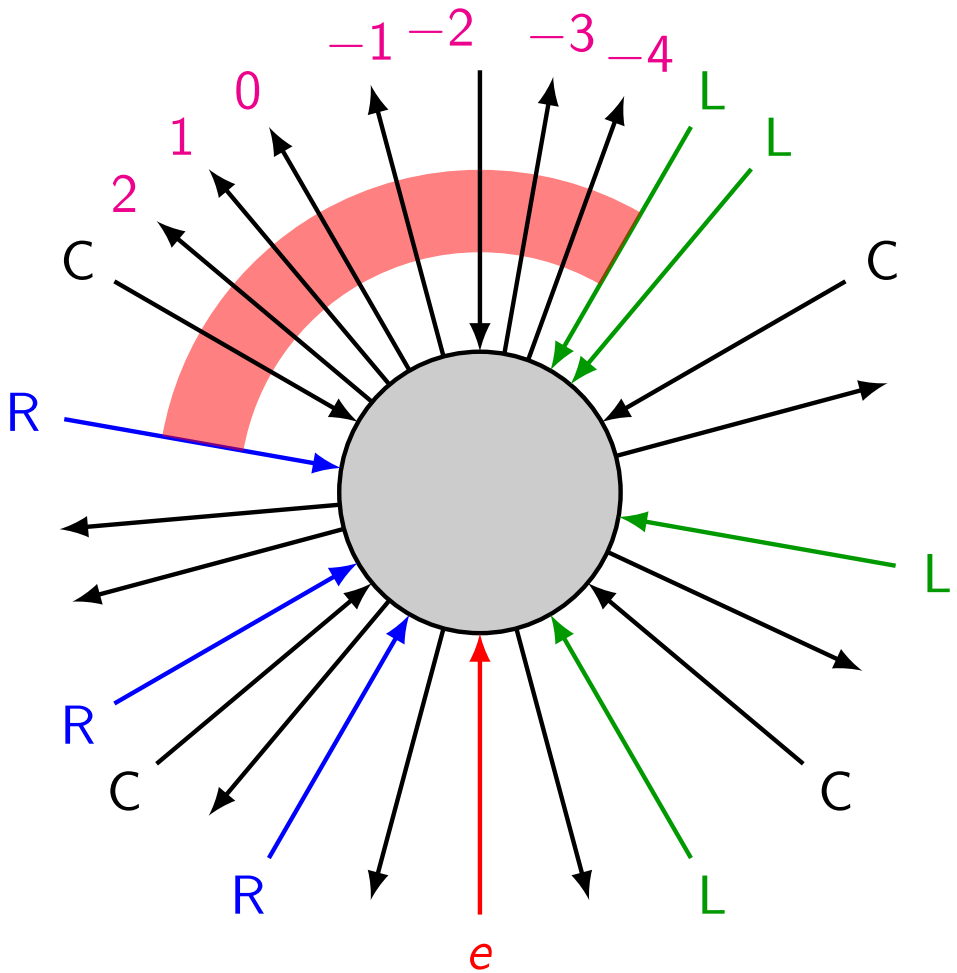

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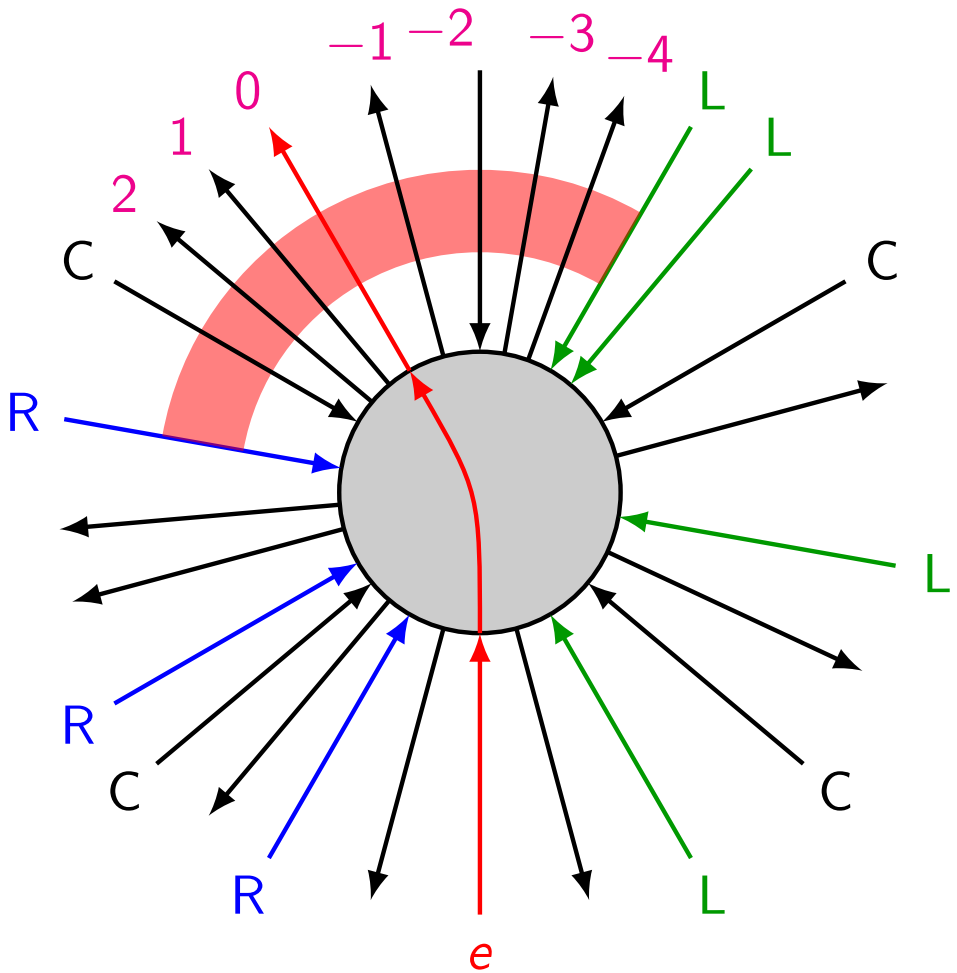

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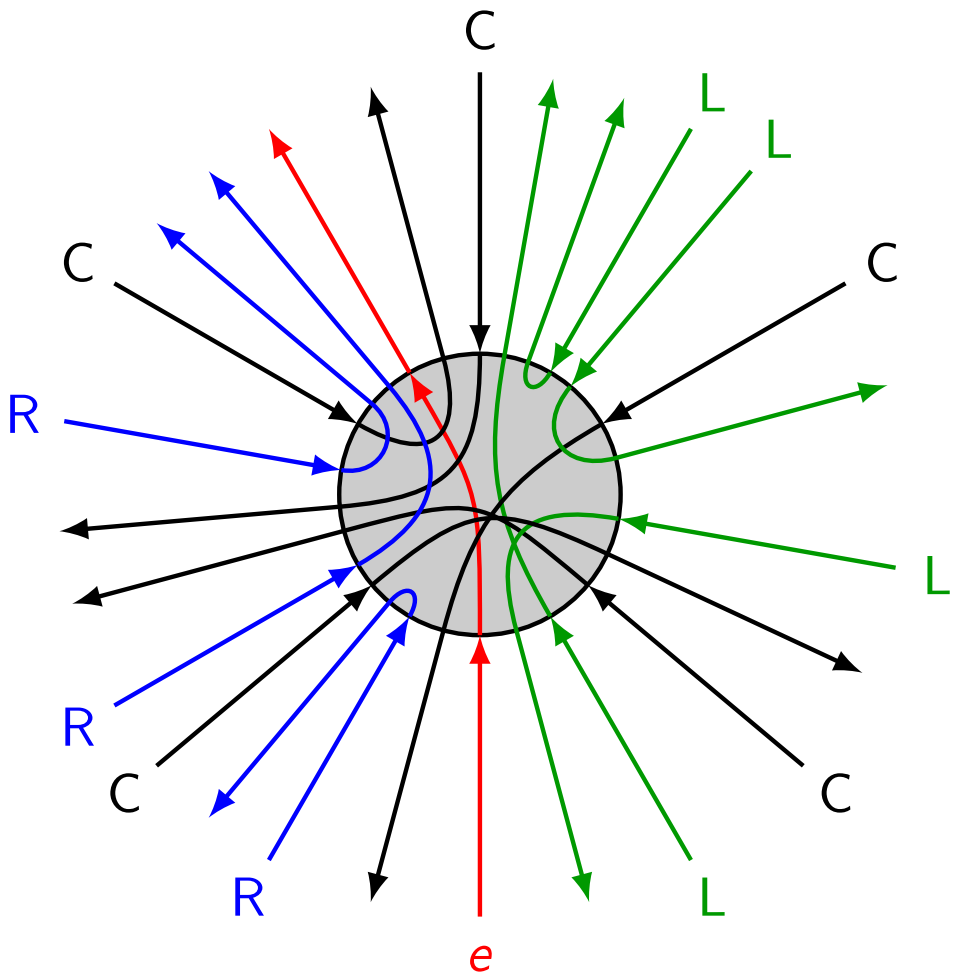

Algorithm

1. Guess the matrices,
2. Route every vertex in increasing order,
3. Check the solution.

Complexity : $O(f(R)^{k^2} n)$

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Open questions

- Fully polynomial algorithm (or weak NP-completeness proof)?
- Directed non-acyclic case?
- Undirected case?

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Other interesting cases

- directed, $G + H$ Eulerian, uncapacitated, $|E(H)|$ fixed? (polynomial for 3, Ibaraki and Poljak)
- directed, $G + H$ Eulerian and planar.
- The round-trip problem.

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Questions?