Maximum edge-disjoint paths problem in k-sums of graphs

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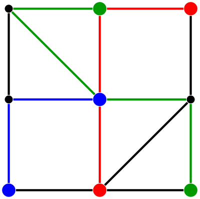
The Edge-Disjoint Paths problem (EDP)

Input:

- a capacitated graph (G, c),
- commodities $(s_i, t_i) \in V(G)^2$, for $i \in [\![1, k]\!]$. Graph $H = (T \subseteq V(G), \{(s_i, t_i)\})$.

Output: Decide whether there is:

- $(P_i)_{i \in [\![1,k]\!]}$, where P_i is a (s_i, t_i)-path in G,
- $|\{i \in [\![1, k]\!] : e \in P_i\}| \leq c(e) \quad$ for all $e \in E(G)$ (disjointness).



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Two optimization problems

EDP is NP-hard.

Maximum Concurrent Integral Flow:
(Min Congestion Int. Flow)

- route every commodity,
- relax disjointness: $|\{i \in [\![1,k]\!] : e \in P_i\}| \leq \frac{1}{\alpha}c(e)$,
- maximize α (minimize $\frac{1}{\alpha}$: the congestion).

Maximum Edge-Disjoint Paths: (Max EDP)

- select a subset of commodities $I \in [\![1,k]\!]$,
- route commodities in I with edge-disjoint paths,
- maximize $|I|$ (or even $w(I)$).

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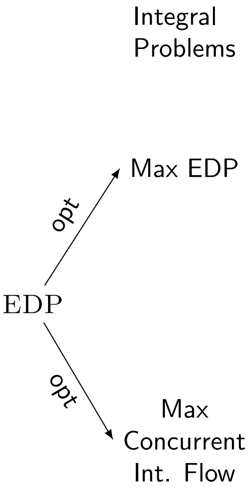
The big picture



Relaxation: multicommodity flows

Let $\mathcal{P}_i$ be the set of all $(s_i, t_i) - \text{paths}$.

IP for Min Congestion Int. Flow:

$$
\begin{array}{lll}
\max & \alpha \quad \text{subject to} & \\
 & \sum_{P \in \mathcal{P}_i} x_P = 1 & \text{(for all } i \in [\![1, k]\!]) \\
 & \sum_{i=1}^{k} \sum_{P \in \mathcal{P}_i, e \in P} x_P \leq \frac{1}{\alpha} \cdot c_e & \text{(for all } e \in E(G)) \\
 & x \in \{0, 1\} &
\end{array}
$$

IP for Max EDP:

$$
\begin{array}{lll}
\max & \sum_{h \in H} z_h \quad \text{subject to} & \\
 & \sum_{P \in \mathcal{P}_i} x_P = z_h \leq 1 & \text{(for all } i \in [\![1, k]\!]) \\
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Relaxation: multicommodity flows

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Max concurrent flow (Max Conc. Flow):

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\begin{aligned}
\max \quad & \alpha \quad \text{subject to} \\
& \textstyle\sum_{P \in \mathcal{P}_i} x_P = 1 && \text{(for all } i \in [\![1, k]\!]) \\
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& x \geq 0
\end{aligned}
$$

Max throughput flow (Max Throughput Flow):

$$
\begin{aligned}
\max \quad & \textstyle\sum_{h \in H} z_h \quad \text{subject to} \\
& \textstyle\sum_{P \in \mathcal{P}_i} x_P = z_h \leq 1 && \text{(for all } i \in [\![1, k]\!]) \\
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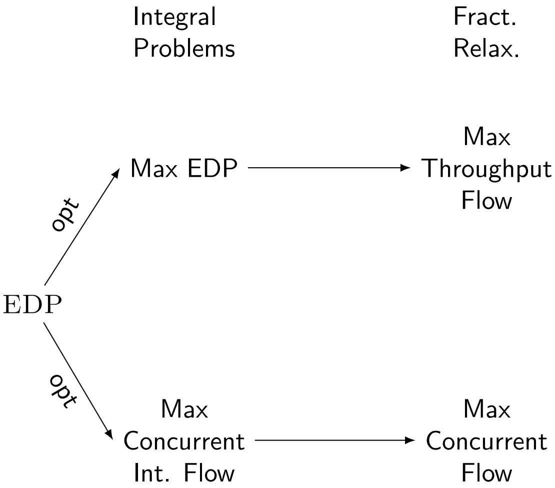

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Cut condition

Inspiration: max-flow-min-cut theorem

EDP: $c(\delta_G(X)) \geq d_H(X)$ for all $X \subseteq V(G)$

Max Conc. Flow: Sparsity of a cut X (with $d_H(X) \neq 0$):

$$\frac{c(\delta_G(X))}{d_H(X)}$$

Max Throughput Flow:
Cardinality of a multicut $E' \subseteq E(G)$:

$$E' \cap E(P) \neq \emptyset \qquad \text{for all } (s_i, t_i)\text{-path } P$$

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Any max-flow-min-cut extension? No!

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The flow-cut gap

EDP, Min Congestion Int. Flow, Sparsest Cut:

NP-hard

Flow-Cut gap: ratio min congestion / sparsest cut
= "How far are we from sufficiency of cut condition."

- $O(\log n)$ in general (Leighton - Rao, Linial - London - Rabinovich),
- $\Omega(\log n)$ in constant degree expanders,
- $O(\sqrt{\log n})$ in planar graphs (Rao),
- $O(1)$ for SP graphs, bounded pathwidth, outerplanar,...
- Conjecture : $O(1)$ in minor-closed families (GNRS conjecture)

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Integral flow – Sparsest cut gap

Flow-cut gap: between fractional flow and sparsest cut.

Gap between integral and fractional flow ?

gap-conjecture (Chekuri, Shepherd, Weibel): there exists f such that for any family of graphs with flow-cut gap β, its integral flow-cut gap is $f(\beta)$.

Examples:

- Series-Parallel graphs (CSW). FC gap $= 2$, integrality gap $= 5$.
- Seymour K_5-free, FC gap $= 1$, integrality gap $= 2$

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The throughput gap

EDP, Max EDP, Min Multicut: NP-hard

Throughput gap: ratio max throughput / min multicut
= “How far are we from sufficiency of cut condition.”

- $\Theta(\sqrt{n})$ in general (Garg - Vazirani - Yannakakis, Chekuri - Khanna - Shepherd),
- $O(1)$ for minor-closed families of graphs (Klein - Plotkin - Rao, Tardos - Vazirani).

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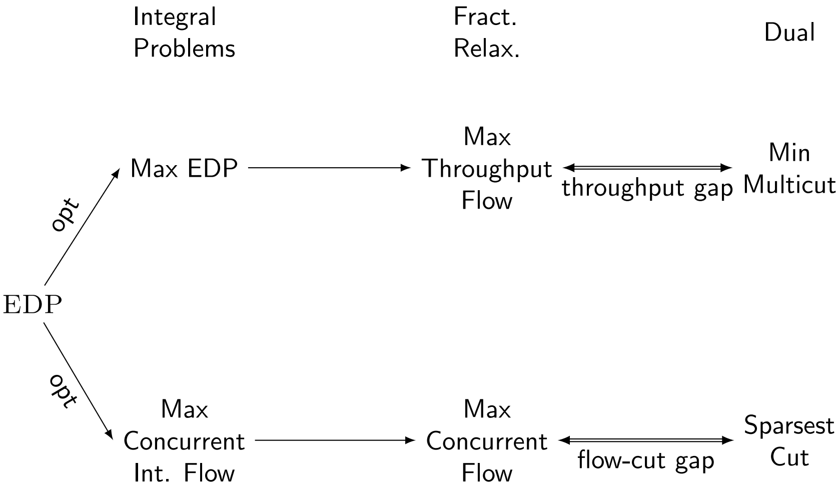
The big picture



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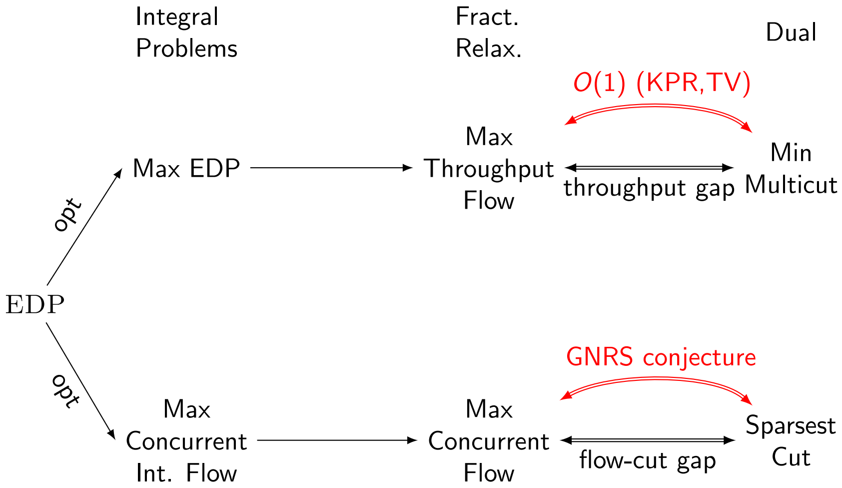

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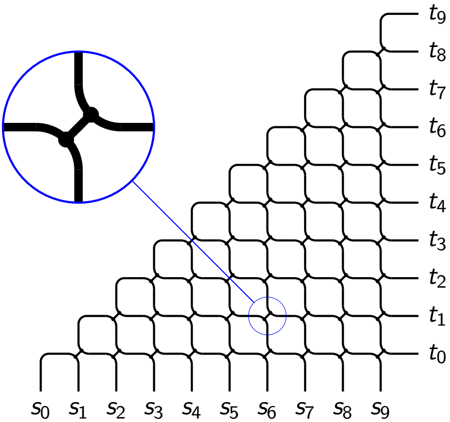
MAX EDP– multicut gap

Throughput gap: between fractional flow throughput and multicut.

Gap between integral and fractional flow ?

Bad $O(\sqrt{n})$ (even in planar graphs):



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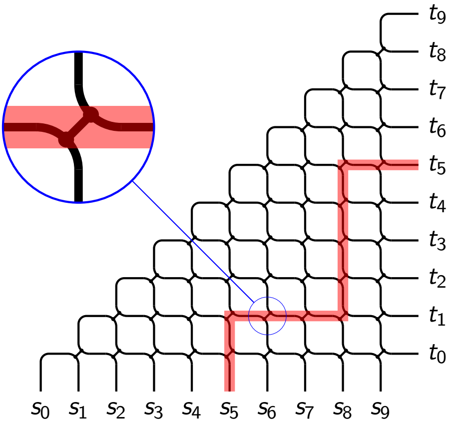

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MAX EDP– **multicut gap**

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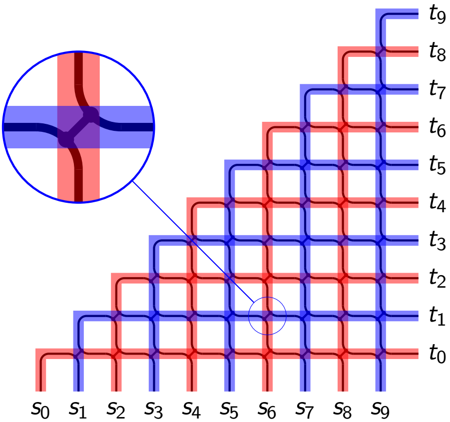

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Closing the integrality gap

$O(\sqrt{n})$ even in planar graphs

Idea: use constant congestion β to get a constant integrality gap. (equivalent: prove constant $\frac{1}{\beta}$-integrality gap)

- constant gap with congestion 2 for planar graphs (Chekuri - Khanna - Shepherd, Seguin-Charbonneau - Shepherd),
- congestion 2 gives $O(\text{polylog}\, n)$ gap for any graph (Chuzhoy - Li).

Our conjecture: Constant gap with congestion 2 for any minor-closed family.

This work: Bounded treewidth and bounded genus.

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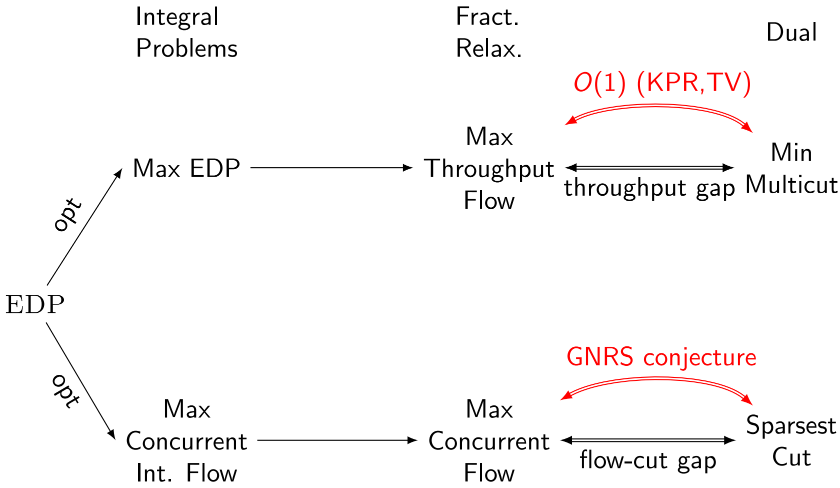

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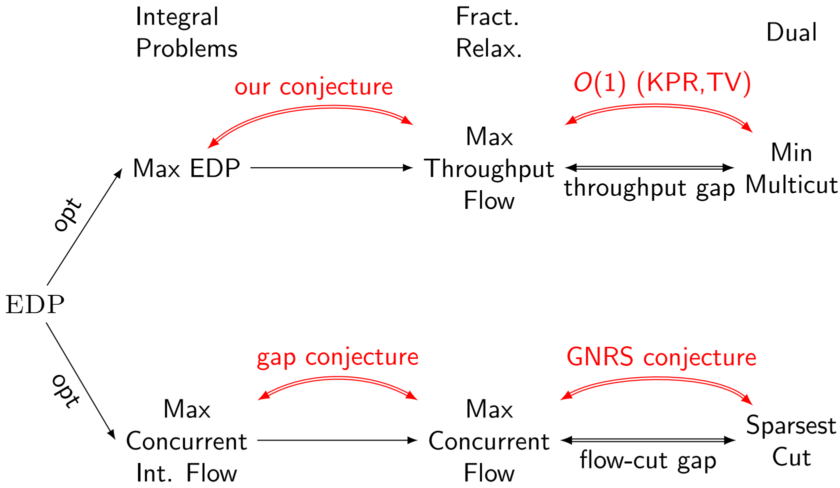

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Minor-closed families

Theorem (Robertson-Seymour)

For any minor-closed family $\mathcal{G}$, there is k such that for all $G \in \mathcal{G}$, G can be obtained by:

- *draw graphs on surfaces of genus $\leq k$,*
- *add $\leq k$ vortices to each of them,*
- *add $\leq k$ apex vertices to each of them,*
- *make k-sums of these graphs.*

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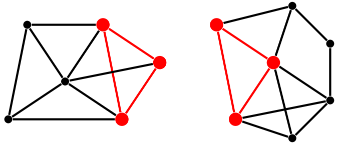
k-sums of graphs

Example of 3-sum:



Treewidth k graph: k-sums of graphs on $k+1$ vertices.

k-sums of graphs

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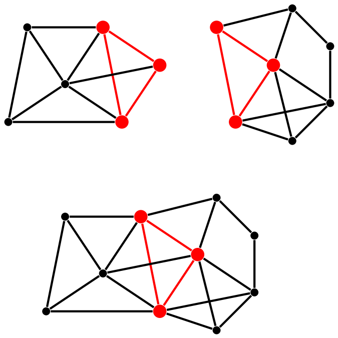

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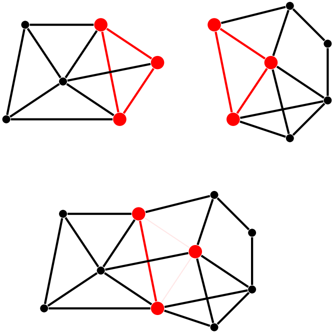

Treewidth k graph: k-sums of graphs on $k+1$ vertices.

Results

Theorem

If $\mathcal{G}$ is minor-closed, with integrality gap α using congestion β, then its closure by k-sums has integrality gap $2^{O(k)}\alpha$ using congestion $\beta + 3$.

Theorem

Graphs of treewidth k have integrality gap $2^{O(k)}$.

Theorem

Graphs of genus g have integrality gap $O(g \log^2(g+1))$ using congestion 3.

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Technical tools

(Partly from Chekuri - Khanna - Shepherd)

1. Moving terminals,

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Moving terminals (I)

$x(v)$: value of flow paths ending at v in fractional optimum.
$=$ marginal value at v

- If there is a flow routing $x(s), s \in S$ to $U \subset V$,
- Then, move the terminals in S to U,
- Up to approximation 5, congestion $+2$:

$$\text{flow } f \text{ in } G \xRightarrow{\text{move terminal}} \text{flow } f' \text{ in } G'$$

$$\text{integral flow } \mathcal{P} \text{ in } G \xLeftarrow{\text{extend paths}} \text{integral flow } \mathcal{P}' \text{ in } G'$$

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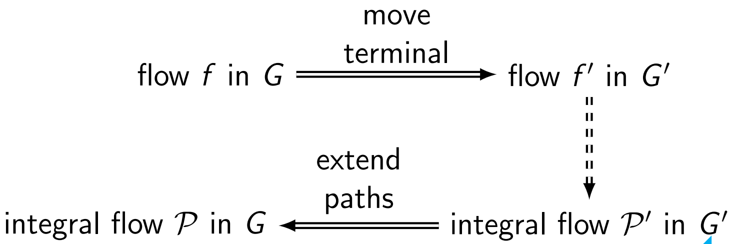
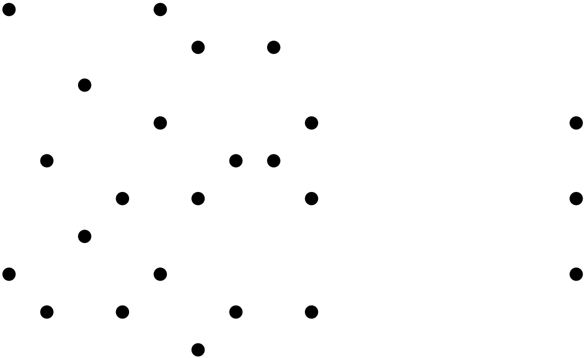


Moving terminals (II)

Cluster, find paths, move terminals and get back.



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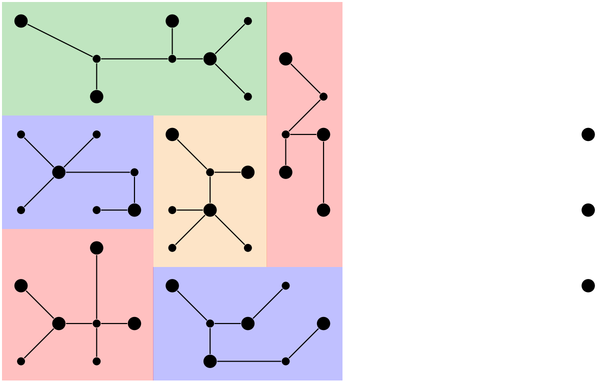

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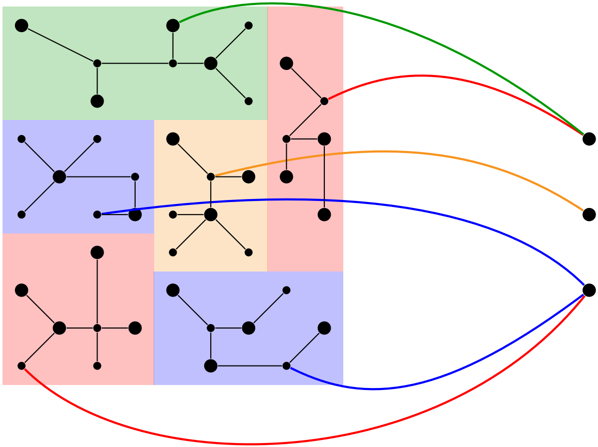

Technical tools

(Partly from Chekuri - Khanna - Shepherd)

1. Moving terminals,
2. Integral sparsifiers,

Integral sparsifiers

Let (G, c) capacitated graph, $S \subseteq V(G)$.

Goal: Get $H = (S, E_H)$ (σ, ρ)-sparsifier with

- Any multiflow between S in G is routable in H with congestion σ,
- Any integral multiflow in H is routable in G with congestion 2.

Using simple splitting-off technique:

Lemma

There is a $(|S|^2, 2)$-sparsifier.

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Technical tools

(Partly from Chekuri - Khanna - Shepherd)

1. Moving terminals,
2. Integral sparsifiers,
3. Routing through bounded number of nodes.

Routing through one node

Lemma

If every flow path of a fractional solution goes through some vertex v, there is an integral solution rounting $\frac{1}{12}$ as much.

Easily extends to small subsets of vertices.

Solves the apex step.