Packing and covering with balls on Busemann surfaces

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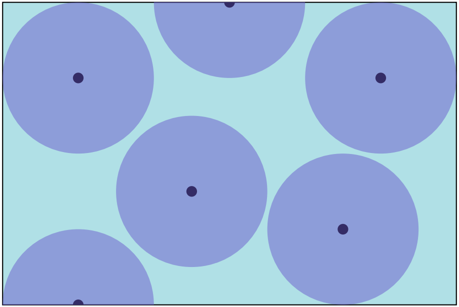
Packing of balls

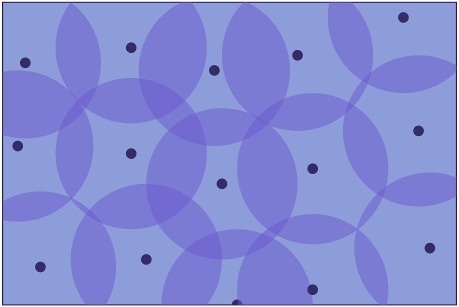
Example of a packing of balls in region S:



Goal: maximize the number of disjoints balls in S.

Covering by balls

Example of a covering by balls of region S:



Goal: minimize the number of balls covering S.

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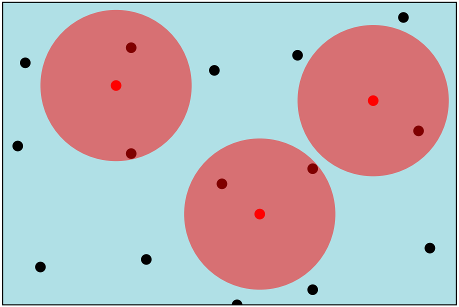
Example of a **set of points** hitting every ball of region S:



Goal: minimize the number of points hitting every ball.

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Packing, covering and hitting sets

General setting:

- X domain (maybe infinite)
- $\mathcal{F} \subseteq 2^X$ subsets of X

Goals:

- find maximum packing $\nu(\mathcal{F})$,
- find minimum covering $\rho(\mathcal{F})$,
- find minimum hitting set $\tau(\mathcal{F})$.

For balls: hitting set = covering

A more specific case:

- X is metric space, S compact in X
- $\mathcal{F} = \{B_\delta(c) \ : \ c \in S\}$ balls of a fixed radius δ (δ-*balls*).

$$x \in B_\delta(c) \iff c \in B_\delta(x)$$
$$\forall x \in S, \exists c \in T, x \in B_\delta(c) \iff \forall x \in S, \exists c \in T, c \in B_\delta(x)$$
$$\text{balls centered at } T \text{ cover } S \iff T \text{ is hitting set for } S$$

Hence: hitting set $\equiv$ covering

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Inequalities

In a packing, each set hits a distinct element of a hitting set.

$$\nu(\mathcal{F}) \leq \tau(\mathcal{F})$$

Question

For which families of sets $\mathcal{F}$ do we have $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$, for some constant c?

Existence of c with $\tau \leq c\nu$

Conjectured:

- families of axis-parallel rectangles (Wegner 1965).

Proved:

- intervals, union of k intervals (Alon 1998),
- some families of subtrees of a tree (Bárány, Edmonds, Wolsey 1986)
- union of k balls in hyperbolic space, using balls in the cover that are slightly larger than in the packing (Chepoi, Estellon 2007),
- axis-parallel rectangles intersecting a monotone curve (Chepoi, Felsner 2013).

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How to prove $\tau \leq c\nu$?

With a primal-dual proof: build packing P and hitting set T with $|T| \leq c|P|$.

When dealing with balls:

- maximal packing of δ-balls $\Longrightarrow$ covering with 2δ-balls,
- then cover each 2δ-balls with c δ-balls,
- this gives a covering, hence a hitting set.

Bounded doubling dimension

A metric space has the *bounded doubling dimension* if for some constant $c \in \mathbb{N}$, for any $\delta > 0$, any 2δ-ball can be covered by c δ-balls.

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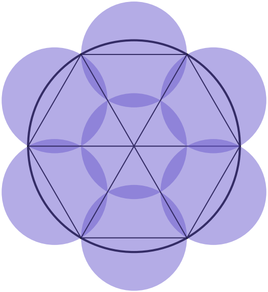
Case of Euclidean convex region in the plane.

Doubling dimension is 7:



Hence $\tau \leq 7\nu$ in convex regions of $\mathbb{E}^2$.

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Definition of Busemann spaces

Let (X, d) be a metric space.
Geodesic path: $\gamma : [0, l] \subset \mathbb{R} \to X$
where $d(\gamma(t), \gamma(t')) = |t' - t|$ (for any $t, t' \in [0, l]$).
Distance function between two geodesic paths $\gamma : [0, l] \to X$, $\gamma' : [0, l'] \to X$:

$$f_{\gamma,\gamma'} : t \in [0, 1] \to d(\gamma(t \cdot l), \gamma'(t \cdot l'))$$

Busemann space

(X, d) is *Busemann* if the distance function between any two geodesic paths is convex.

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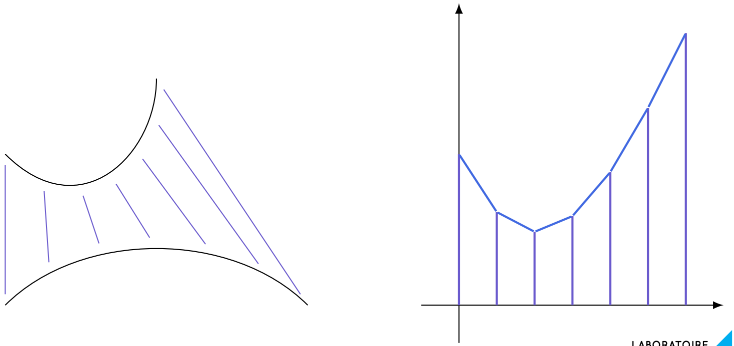
Illustration of the distance function.

Distance function between two geodesic paths $\gamma : [0, l] \to X$, $\gamma' : [0, l'] \to X$:

$$f_{\gamma,\gamma'} : t \in [0, 1] \to d(\gamma(t \cdot l), \gamma'(t \cdot l'))$$



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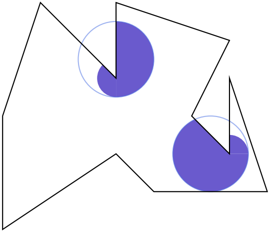
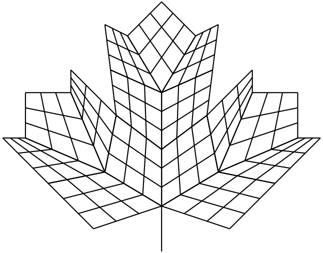
Examples of Busemann spaces

- Euclidien space,
- hyperbolic *n*-dimensional spaces,
- Strictly convex normed vector spaces,
- CAT(0) spaces, non-positively curved spaces

Examples of Busemann surfaces

- simple polygons of the Euclidean plane,
- planar contractile complexes non-positively curved: the sum of angles around each vertex is at least 2π.





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Properties of Busemann spaces

- unique geodesic between any two points,
- an everywhere-locally-geodesic curve is a geodesic,
- balls are convex,
- geodesic paths vary continuously with their endpoints,

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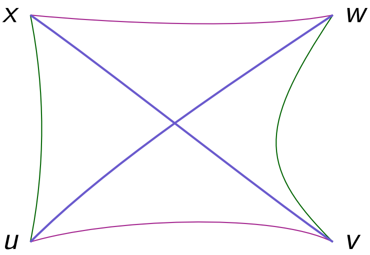
Properties of Busemann surfaces

In dimension 2:

- contractile (no holes),
- extensible: can be extended as a convex region of Busemann surface without boundary,
- triangles are convex, Peano and Pasch axioms, Helly property, . . .
- quadrangle inequality: for any quadrangle u, v, w, x,

$$d(u,w) + d(v,x) \geq \max\{d(u,v) + d(w,x), d(u,x) + d(v,w)\}$$



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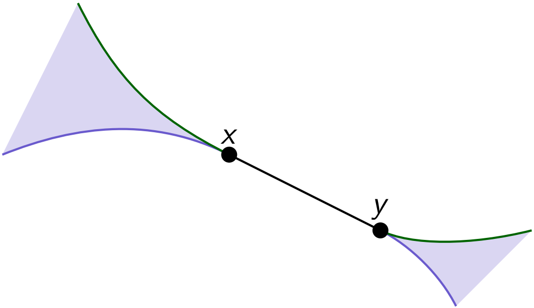
Bad properties of Busemann surfaces

Different from Euclidean plane:

- arbitrary many lines through two given points,
- two lines may be tangent: intersecting but not crossing.



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Our main result

Theorem

$\tau \leq 19\nu$ for balls of radius δ in a compact region of a Busemann surface.

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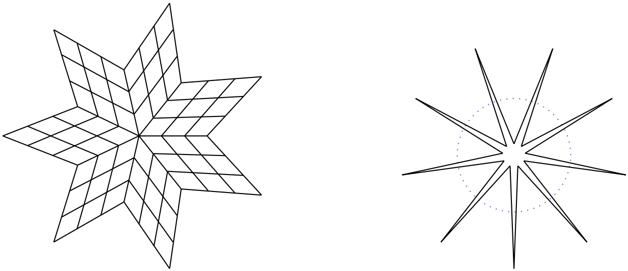
Proof by bounded doubling dimension?



Doubling dimension is not bounded.

Alternative plan: weak doubling property

Weak doubling property

(X, d) has the *weak doubling property* if there is a constant $c > 0$ such that: for any $\delta > 0$, there is $p \in X$ such that $B_{2\delta}(p)$ can be covered by at most c δ-balls.

Instead of asking for every point, we ask only for one point.

Detailed plan

Proposition

For a compact S in a metric space (X, d) with the weak doubling property with constant c, $\rho(S) \leq c\nu(S)$.

Proposition

Busemann surfaces have the weak doubling property with $c \leq 19$.

Lemma

Busemann surfaces have the bounded simplex-ball covering property: Any region of diameter 2δ can be covered with a constant number of δ-balls.

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Proof: weak doubling property $\implies \rho \leq c\nu$

Primal-dual algorithm:

- Pick a vertex p as in the weak doubling property.
- add $B_\delta(p)$ in packing,
- add at most c balls covering $B_{2\delta}(p)$ in covering.
- remove $B_{2\delta}(p)$ from S,
- and repeat.

(hard part: dealing with open/close balls.)

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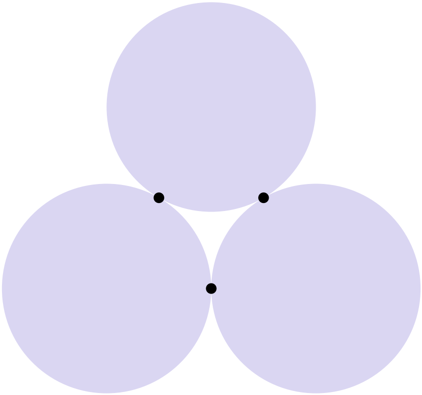
Bounded simplex-ball covering property

S as diameter $2\delta \implies \tau \leq c$



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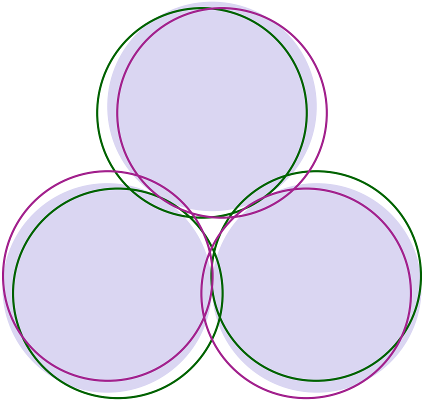
Bounded simplex-ball covering property

S as diameter $2\delta \implies \tau \leq c$



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In Busemann surface

Proposition

Bounded simplex-ball covering property true with $c = 3$.

Proof:

- If any three balls contain a common point, use Helly property, hitting set of size 1.
- Otherwise, find triangle $\Delta(x, y, z)$ maximizing the perimeter of its *inner triangle* $\Delta(x', y', z')$. Then $\{x', y', z'\}$ is a hitting set.

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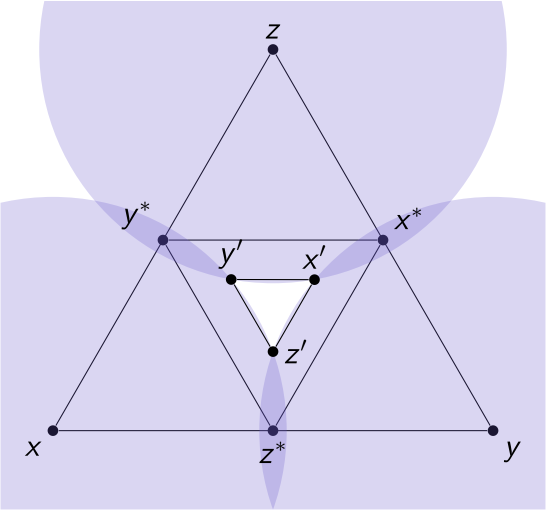
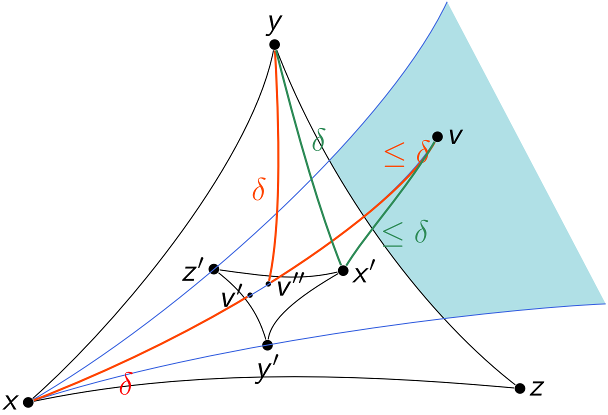
The critical triangle of $\Delta(x, y, z)$



Proving every point v is covered

This is one among four cases:



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Proof of weak doubling property

Proposition

There is a point v such that $B_{2\delta}(v)$ can be covered by at most 19 balls of radius δ.

- choose u, v a diametral pair,
- cover $B_{2\delta}(v)$, by processing each side of $[u, v]$ separately,
- cover each of three regions separately.

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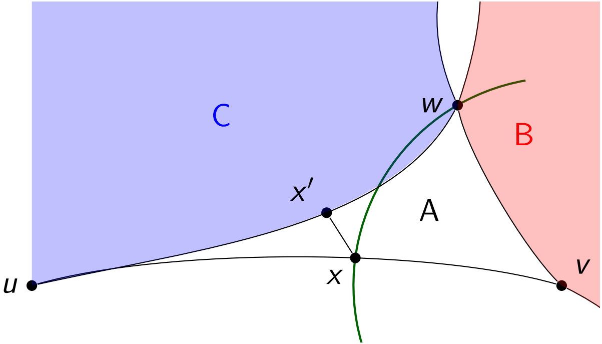
The three regions



- A can be covered with 4 balls,
- prove that $B \cap B_{2\delta}(v)$ has diameter at most 2δ,
- prove that $C \cap B_{2\delta}(v)$ has diameter at most 2δ

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Open questions

- Find a polynomial-time algorithm for simple polygons, to construct packing and covering of similar sizes.
- Extension to polygons with holes? Does the weak doubling property still hold?
- Extension to any 2-dimensional Busemann space? In particular CAT(0) square complexes.

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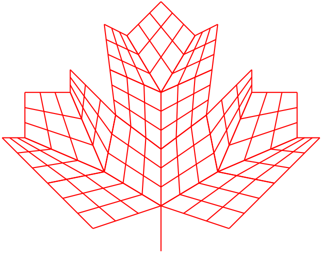
The end.

Thank you!



Questions?

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