Mader-Mengerian Graphs

Vincent Jost[1], Guyslain Naves[2]

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[1]CNRS, LIX, École Polytechnique, Palaiseau

[2]McGill University, Montréal

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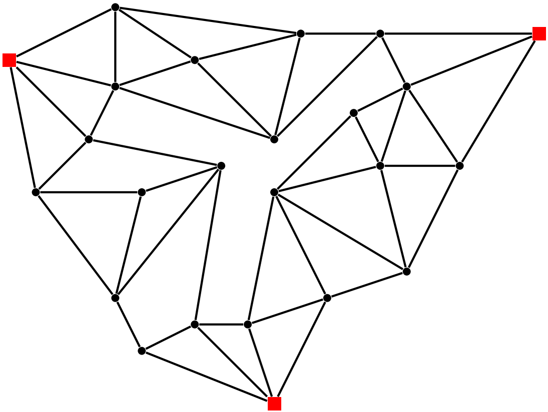
Multiway cuts and disjoint $\mathcal{S}$-paths



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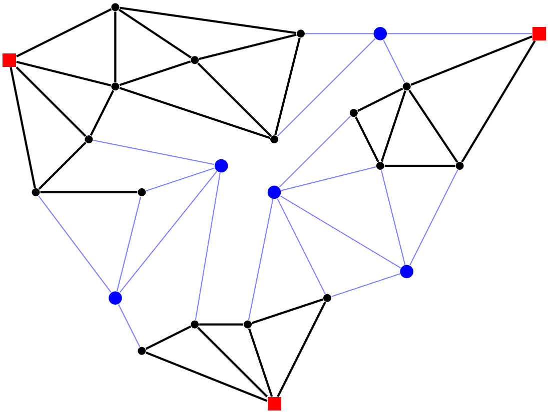

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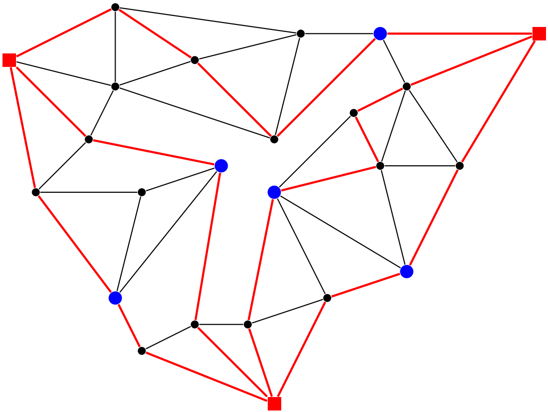

Two problems...

Problem (Min Blocker of T-paths)

Given a graph G, weights $w : V \to \mathbb{N}^+$, and an independant set T, find a subset of vertices S of minimum weight such that each T-path intersects S.

Problem (Max disjoint T-paths)

Given a graph G, weights $w : V \to \mathbb{N}^+$, and an independant set T, find a maximum w-packing of T-paths

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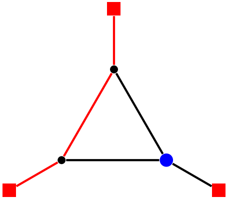
. . . but no Min-Max theorem.



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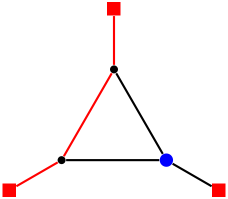

Theorem (Mader, 1978)

Good characterization for the maximum number of T-paths.

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Our goal

Objective:

Find the graphs admitting a Min-Max theorem
(more exactly a TDIness property).

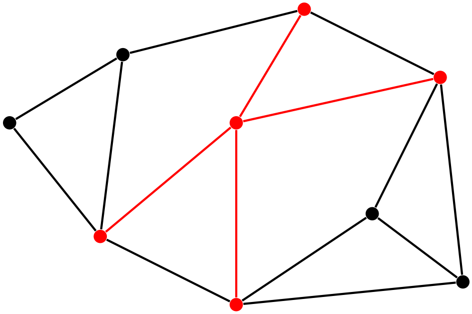
- characterization (by excluded minors),
- algorithm for recognition,
- and algorithms to find cuts and paths.

Vertex-minors

(not the usual definition of vertex-minors)

We define two operations:

- *vertex deletion*,
- *vertex contraction*.



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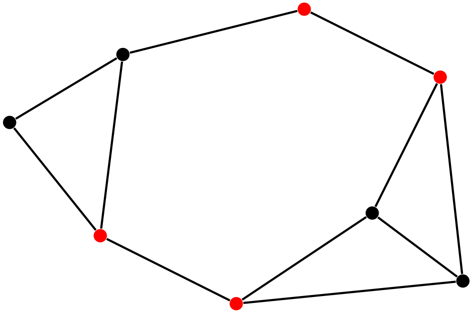

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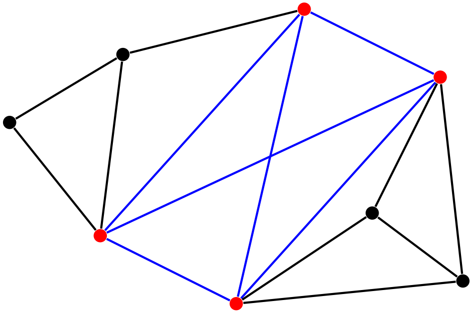
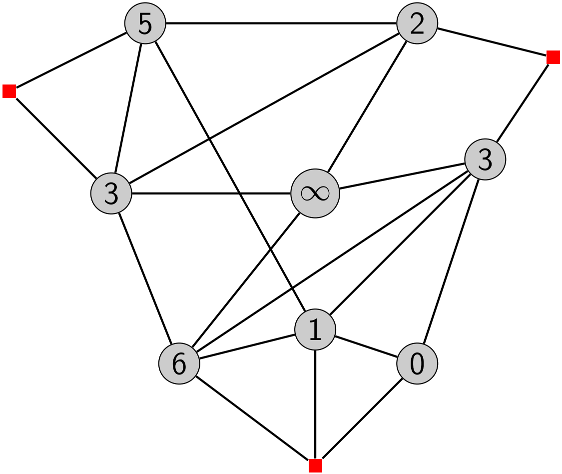


Why these operations?



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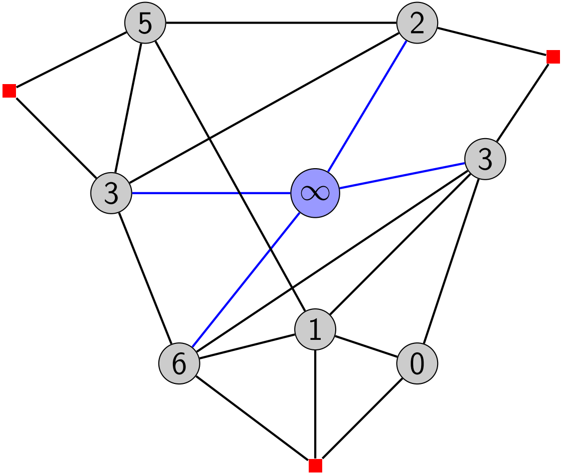

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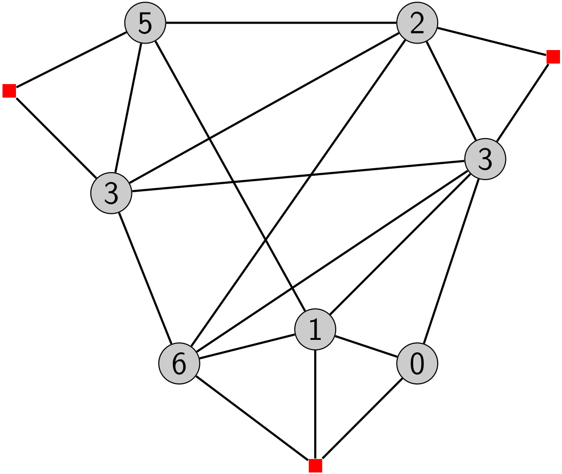

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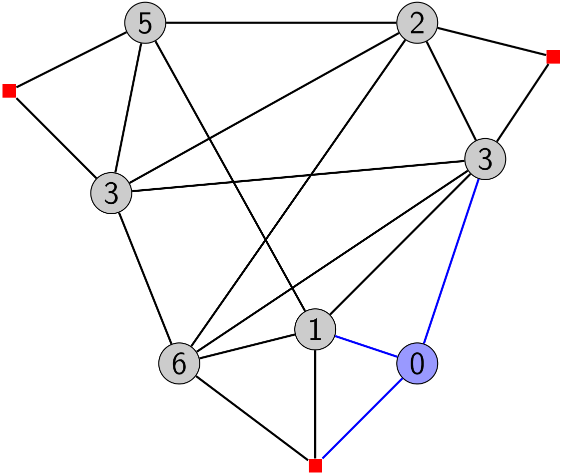

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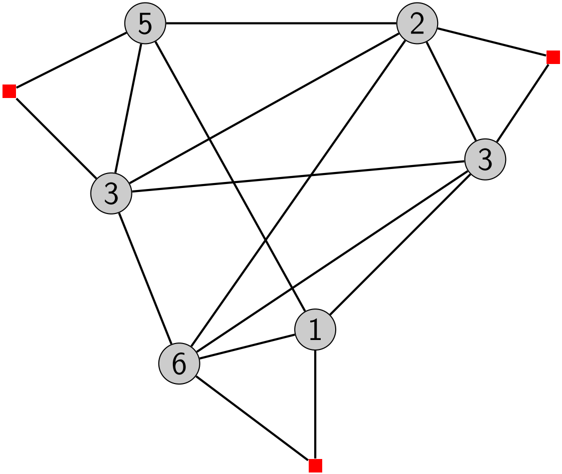


Immediate consequence

Proposition

The class of graphs for which the following system is TDI:

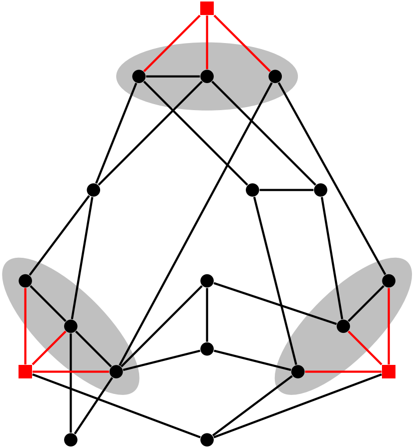
$$x(P) \geq 1 \quad \textit{for all } T\textit{-path } P$$

is closed under vertex-minor operations.

TDIness of $x(P) \geq 1$

When is the following system TDI?

$$x(P) \geq 1 \quad \text{for all } T\text{-path } P$$
$$x \in \mathbb{R}_+^V$$



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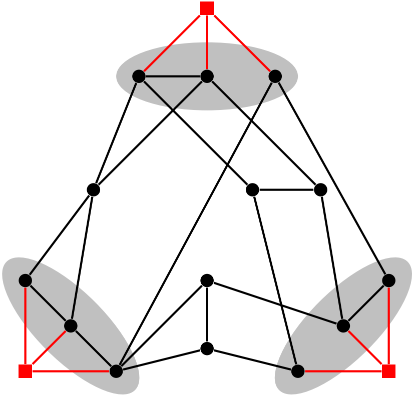

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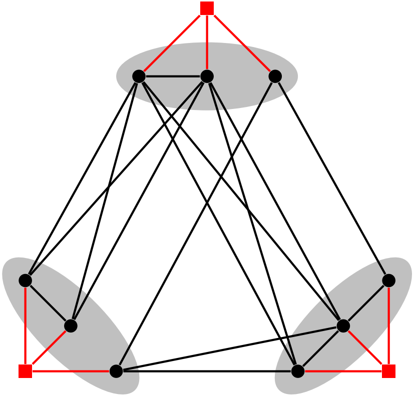

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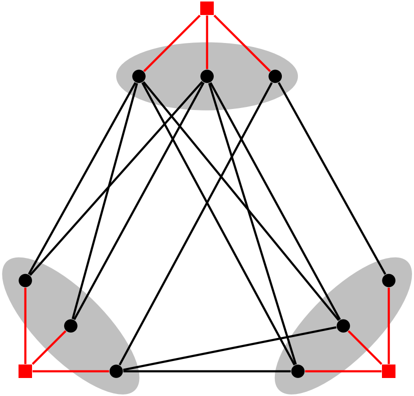

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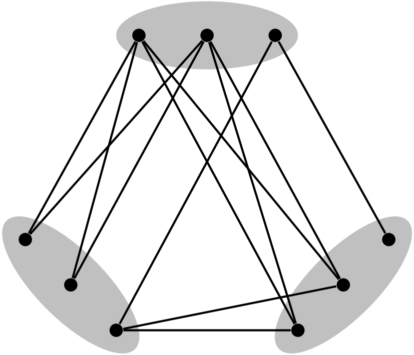

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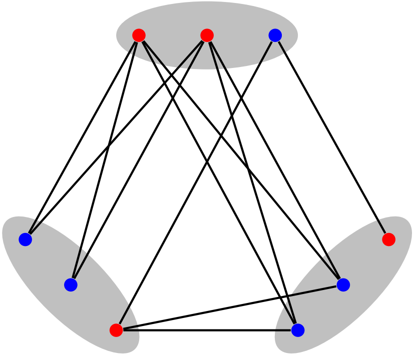

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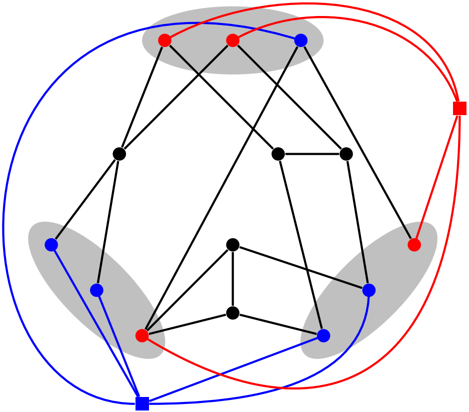
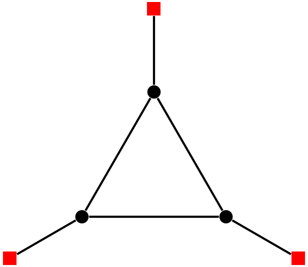


Algorithm

1. Remove nodes adjacent to two terminals.
2. Remove components of $G - (T \cup N(T))$ adjacent to only $N(t)$ for some $t \in T$.
3. Contract the other components.
4. Remove the edges in $N(t)$ for all $t \in T$.
5. If the graph obtained is bipartite, use Menger theorem to find a maximum packing of paths and a minimum multiway cut.
6. If the graph is not bipartite, the system is not TDI.

Non-bipartite auxiliary graph.



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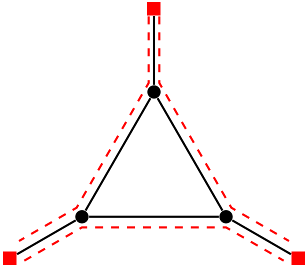

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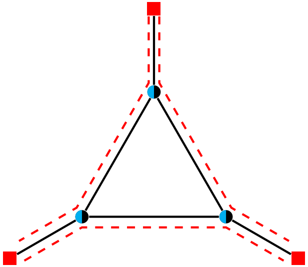

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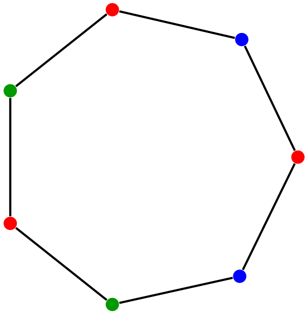

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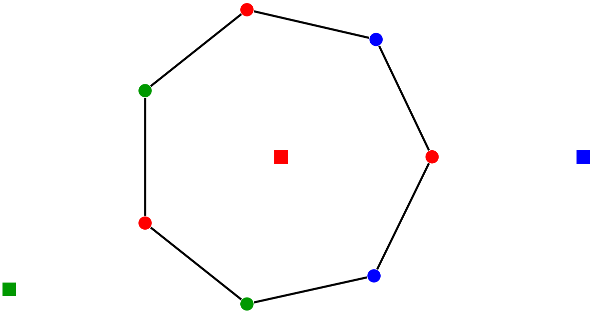

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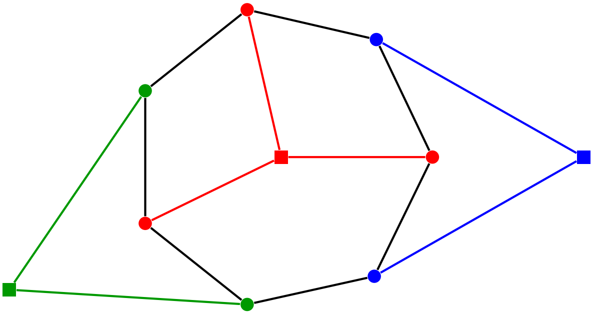

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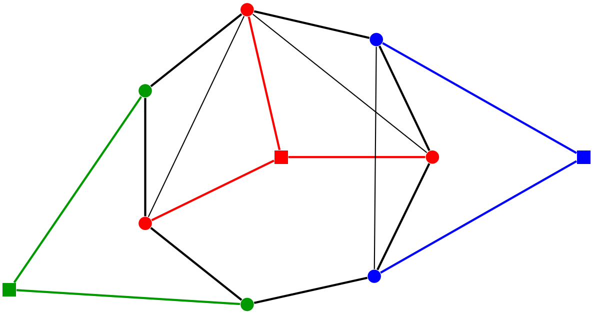

Characterization

Theorem

Let G be a graph. Exactly one is true:

- *the system $x(P) \geq 1$ for every T-path is TDI for any independant set T,*
- *or G contains a bad vertex minor.*

Moreover, in the first case:

algorithm based on min flow to solve
multiway cut and path packing.

Mader-Mengerian graphs

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Recognizing Mader-Mengerian graphs

Checking bipartiteness for any T?

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Theorem

A graph is Mader-Mengerian iff its auxiliary graph is bipartite for every independant set T with $|T| = 3$.

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Checking bipartiteness for any T?

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Polynomial algorithm for recognition.

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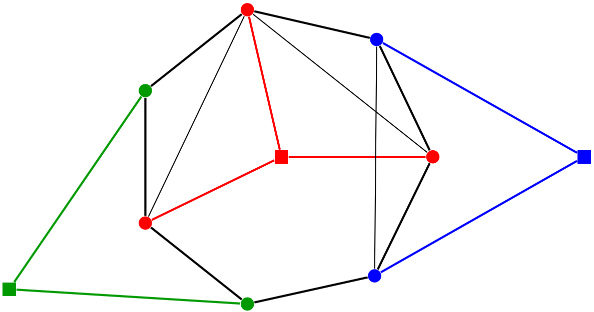
Non-bipartite auxiliary graph.



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The structure of bad graphs

Bad graph:

- odd cycle properly colored,
- one terminal for each color,
- only monochromatic chords.

Lemma

Every color is adjacent to at most two other colors, or G has a net.

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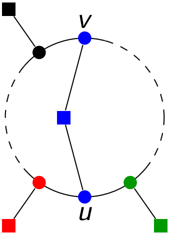

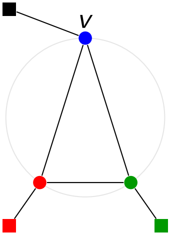

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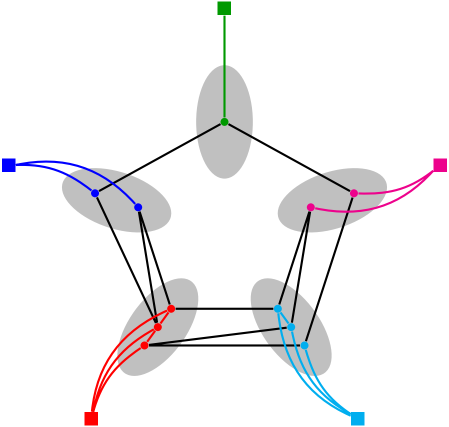
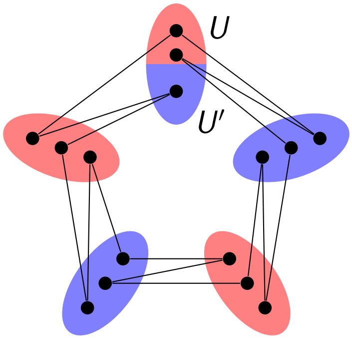


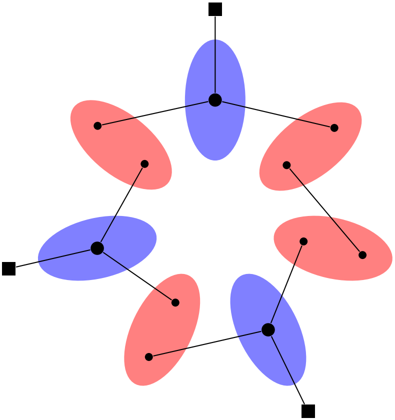
Properties of the colors

Lemma

- *there is an odd number of colors,*
- *each color class is adjacent to exactly two other color classes,*
- *every color class contains a bicolored vertex.*



At most 5 colors



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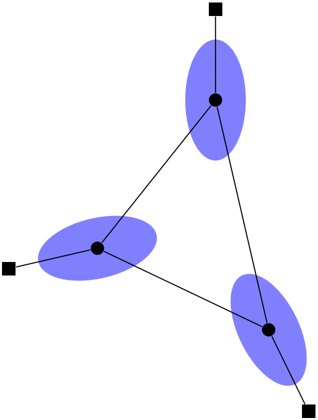
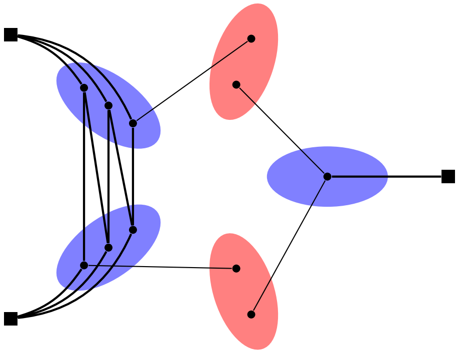


At most 3 colors



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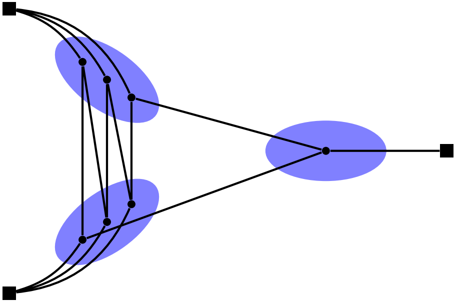
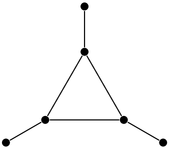
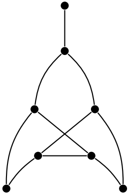


More about Mader-Mengerian graphs

- AT-free graphs are Mader-Mengerian,
- Infinite family of forbidden vertex-minors,
- graphs without rockets and nets by vertex-minor + edge-contraction are Mader-Mengerian:





TDIness for (G, T)

For which (G, T) the system $x(P) \geq 1$ for all T-path P is TDI ?

Theorem

It is TDI for (G, T) iff it is TDI for (G, T') for every $T' \subset T$, $|T'| = 3$.

- Same proof, with a strenghtening of the minor operations.
- Exact characterization by excluded minors.

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Thank you!