Maximum Flows on Disjoint Paths

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Flows, disjoint paths and weights

- Flow on (G, u) = packing of *weighted* paths,
- Edge-disjoint paths problem (EDP)

  = special case $u = 1$

  = packing of *edge-disjoint unweighted* paths,
- How about packing *edge-disjoint weighted* paths?

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Our problem

Disjoint Weighted Flow problem:

Find a maximum flow decomposable in (weighted) edge-disjoint paths

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DISJOINT WEIGHTED FLOW problem:

Find a maximum flow decomposable in (weighted) edge-disjoint paths

Input: $G = (V, E)$, $s, t \in V$, and $u : E \to \mathbb{N}$.
Output: A set $\mathcal{P}$ of edge-disjoint (s, t)-paths.
Objective: maximize $\sum_{P \in \mathcal{P}} \min_{e \in P} u(e)$

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k-splittable flows

Some related work: k-splittable flows

k-splittable flow:

flow decomposable in k (non-necessarily disjoint) paths.

Theorem (Baier, Kölher, Skutella 2005)

2-approximation algorithm for k-splittable flow

Theorems

Theorem

$\Theta(\log n)$ *upper and lower bound for approximability of* DISJOINT WEIGHTED FLOW, *even in planar undirected graphs.*

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Theorems

Theorem

$\Theta(\log n)$ *upper and lower bound for approximability of* Disjoint Weighted Flow, *even in planar undirected graphs.*

Theorem

$\Theta(\log k)$ *upper and lower bound for approximability of* Disjoint Weighted Flow *with at most k paths, even in planar undirected graphs.*

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Upper bound

Theorem

There is an $O(\log n)$ approximation algorithm for the DWF problem.

Upper bound

Theorem

There is an $O(\log n)$ approximation algorithm for the DWF problem.

- Round and scale down the weight function (lose a constant factor).
- Solve $\log n$ flow problems, keep the best solution.
- Optimal solution is less than the sum of the $\log n$ flows.

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Lower bound

Theorem

It is $\Omega(\log n)$*-hard to approximate the planar undirected DWF problem (unless P=NP).*

Lower bound

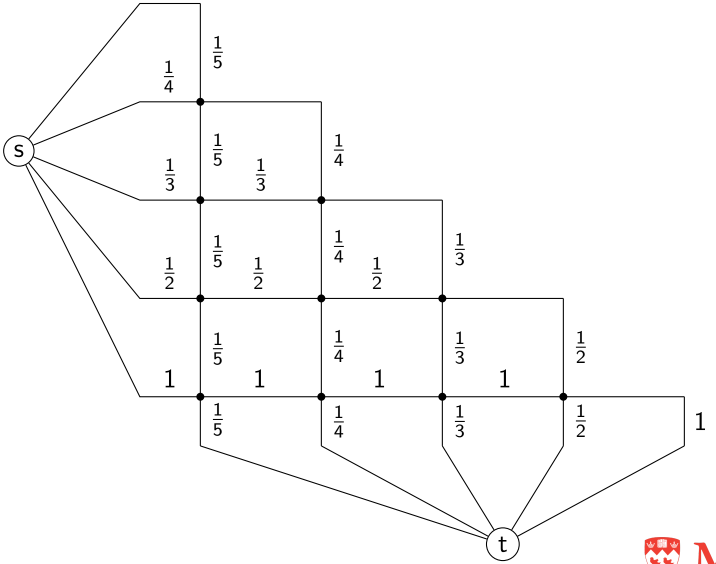
Theorem

It is $\Omega(\log n)$-hard to approximate the planar undirected DWF problem (unless P=NP).

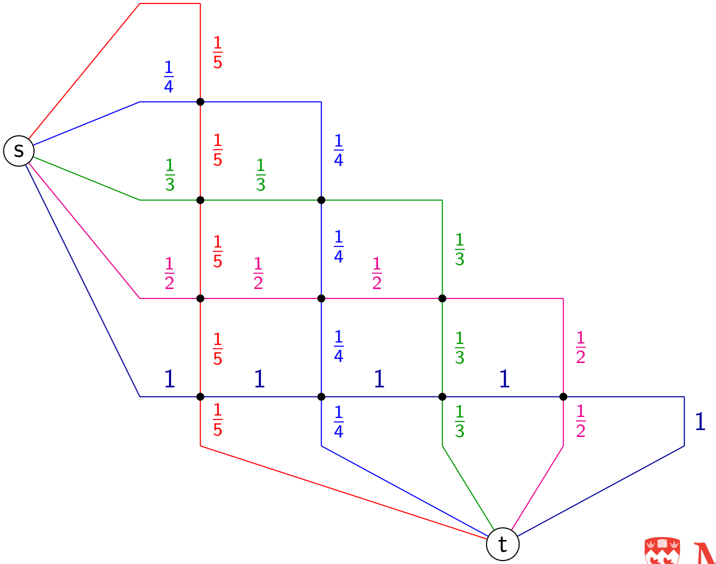
- Define Half-Grid$_n$ whose maximum DWF is $\log n$.
- Take an instance H of a particular NP-hard problem.
- Replace every vertex of Half-Grid$_n$ by H.
- max DWF of the resultant graph is either 1 or $\log n$, depending on the existence of a solution for H.

(see Guruswami, Khanna, Rajaraman, Shepherd, Yannakakis 2003)

weighted Half-Grid$_5$



weighted Half-Grid$_5$



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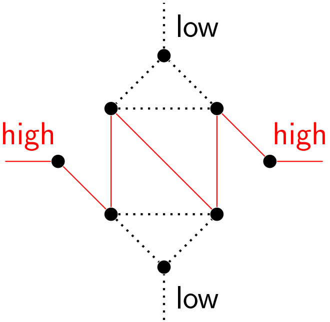
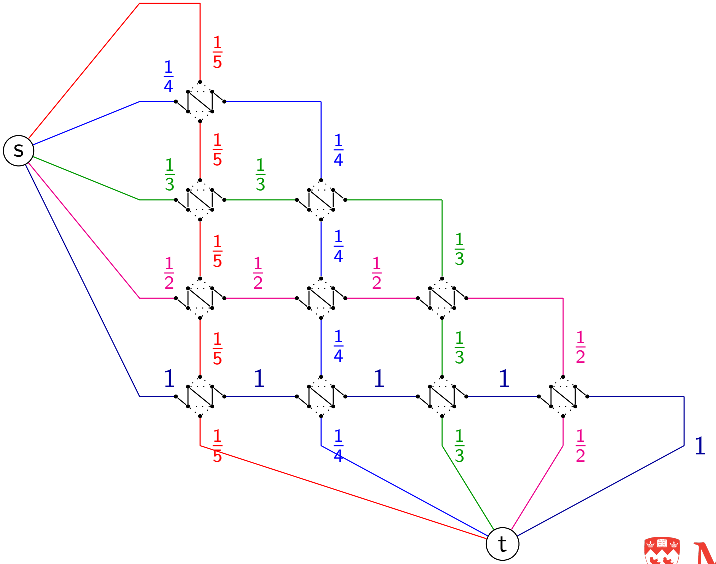
Replacing the vertices

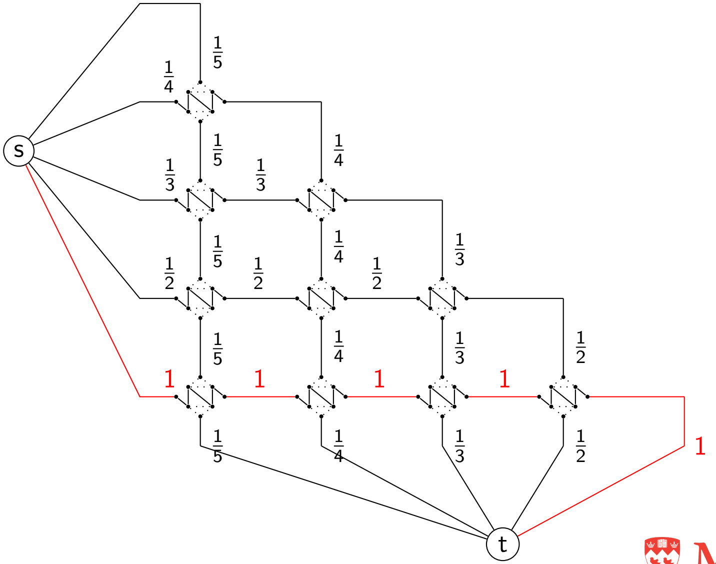
What happens when replacing the non-terminal vertices by this gadget?



Replacing the vertices



Replacing the vertices



2-paths problem

Two edge-disjoint paths problem, with black and red edges, and one of the paths must use only red edges.

- red edges: high weight,
- black edges: low weight.
- NP-complete with only red edges (Fortune, Hopcroft, Wyllie)

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2 Edge-Disjoint Weighted Paths

Given G a graph, $s_a, t_a, s_b, t_b \in V(G)$ and $B \subseteq G(E)$, are there an (s_a, t_a)-path P_a and an (s_b, t_b)-path P_b, edge-disjoint, with $E(P_b) \subseteq B$?

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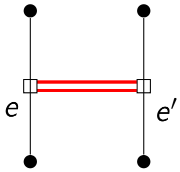
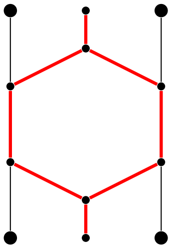
Sketch of the NP-hardness proof (I)

Finding an (s,t)-path P with constraints:

$$P \text{ uses at most one of } e, e'$$

(Fortune, Hopcroft, Wyllie 1980)





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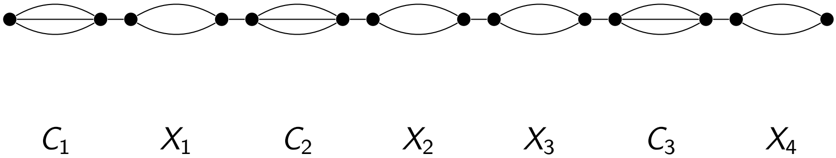
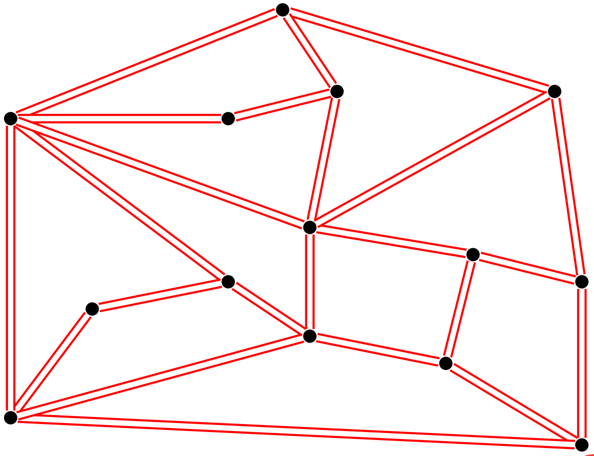
Sketch of the NP-hardness proof (II)

$$(X_1 \vee X_2 \vee \overline{X_3}) \wedge (X_1 \vee X_2 \vee \overline{X_3}) \wedge (\overline{X_1} \vee X_3 \vee \overline{X_4})$$



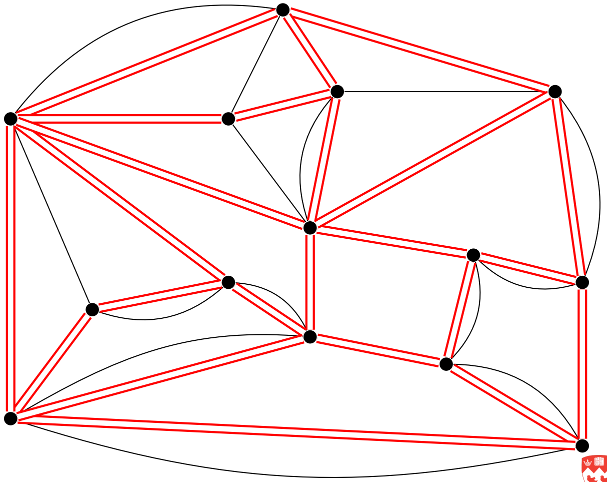
Sketch of the NP-hardness proof (III)

Getting the planarity: from PLANAR-3-SAT



Sketch of the NP-hardness proof (III)

Getting the planarity: from Planar-3-Sat



Perspectives

Weighted Multicommodity Flows:

- Special case: EDP, approximability $\alpha = O(\sqrt{m})$,
- with our method, $\alpha \log n$ approximation.

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- Special case: EDP, approximability $\alpha = O(\sqrt{m})$,
- with our method, $\alpha \log n$ approximation.

Bi-criteria:

- Adding some congestion c,
- as in the recent developments for EDP.