# Non-Redistributive Second Welfare Theorems

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**Abstract.** The second welfare theorem tells us that social welfare in an economy can be maximized at an equilibrium given a suitable redistribution of the endowments. We examine welfare maximization without redistribution. Specifically, we examine whether the clustering of traders into $k$ submarkets can improve welfare in a linear exchange economy. Such an economy always has a market clearing $\varepsilon$-approximate equilibrium. As $\varepsilon \to 0$, the limit of these approximate equilibria need not be an equilibrium but we show, using a more general price mechanism than the reals, that it is a "generalized equilibrium". Exploiting this fact, we give a polynomial time algorithm that clusters the market to produce $\varepsilon$-approximate equilibria in these markets of near optimal social welfare, provided the number of goods and markets are constants. On the other hand, we show that it is NP-hard to find an optimal clustering in a linear exchange economy with a bounded number of goods and markets. The restriction to a bounded number of goods is necessary to obtain any reasonable approximation guarantee; with an unbounded number of goods, the problem is as hard as approximating the maximum independent set problem, even for the case of just two markets.

## 1 Introduction

The fundamental theorems of welfare economics are considered "the most remarkable achievements of modern microeconomic theory" [23] and are the "central set of propositions that economists have to offer the outside world - propositions that are in a real sense, the foundations of Western capitalism" [11]. Informally, they state (under certain conditions that we discuss later)

**First Fundamental Welfare Theorem.** A competitive equilibrium is pareto efficient.

**Second Fundamental Welfare Theorem.** Any pareto efficient solution can be supported as a competitive equilibrium.

The First Welfare Theorem is widely viewed as "a mathematical statement of Adam Smith's notion of the invisible hand leading to an efficient allocation" [31]. The Second Welfare Theorem implies that we can separate out issues of economic efficiency from issues of equity. Specifically, by redistributing the initial endowments (by lump-sum payments), a set of prices exists that can sustain any pareto solution. This second theorem has "fundamental implications for how we think about economic organization" [32] and is "arguably the theoretical result that has had the most dramatic effect on economic thinking" [10]. Despite this, "much of public economics takes as its starting point the rejection of the practical value of the second theorem" [26]. Why this discrepancy? To understand this, note that lump-sum transfers are theoretically considered a very desirable form of taxation as they do not distort incentives within the pricing mechanism. However this is essentially accomplished by a massive distortion of the initial market! Moreover, these are *personalized* liabilities which in turn can be viewed as an extremely unfair form of taxation in that they don't depend upon the actions or behaviours of the agents, and are impractical for a myriad of implementational and political reasons (see, for example, [4], [5], [26] and [25]).

This observation motivates our work. Can the market mechanism be used to sustain pareto allocations without redistribution? In particular, suppose that without redistribution a single market leads to low social welfare (or even has no competitive equilibrium at all). In these circumstances, can the market mechanism still be used to produce an allocation of high social welfare? We address this question under the classical model of exchange economy, and show that indeed this can often be achieved provided the single market can be clustered into submarkets.

### 1.1 The Exchange Economy.

We consider the classical model of an *exchange economy* – an economy without production. We have $n$ traders $i \in \{1, 2, \ldots, n\}$ and $m$ goods $j \in \{1, 2, \ldots, m\}$. (To avoid any ambiguity between traders and good we will often refer to good $j$ as good $g_j$). Each trader $i$ has an initial endowment $\mathbf{e_i} \in \mathbb{R}^m_+$, where $e_{ij}$ is the quantity of good $j$ that she owns, and a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$. The traders have no market power and so are *price-takers*. Given a set of prices $\mathbf{p} \in \mathbb{R}^m_+$, where $p_j$ is the price of good $j$, trader $i$ will demand the best bundle she can afford, that is, $\text{argmax}_{x_i}\, u_i(\mathbf{x_i})$ *s.t.* $\mathbf{p} \cdot \mathbf{x_i} \leq \mathbf{p} \cdot \mathbf{e_i}$. These prices and demand bundles form an *Walrasian (competitive) equilibrium* if all markets clear: demand does not exceed supply for any good $j$. That is, $\sum_i x_{ij} \leq \sum_i e_{ij}$. In this paper we focus on the basic case of linear utility functions – the *linear exchange model*. Here the function $u_i(.)$ can be written as $u_i(\mathbf{x}_i) = \sum_{j=1}^m u_{ij}\, x_{ij}$ where $u_{ij} \geq 0$ is the utility per unit that trader $i$ has for good $j$. (We denote by $\mathbf{u}_i$ the vector of utility coefficients for trader $i$.)

### 1.2 The Fundamental Welfare Theorems.

An allocation is *pareto efficient* if there is no feasible allocation in which some trader is strictly better off but no trader is worse off. The first welfare theorem states that any Walrasian equilibrium is pareto efficient. It holds under very mild conditions, such as monotonic utilities or non-satiation. Clearly for this result to be of interest, though, we need this economy to possess Walrasian equilibria. In groundbreaking work, Arrow and Debreu [3] showed that this is indeed the case, under certain conditions such as concave utility functions and positive endowments.[1] Interestingly, equilibria need not exist even in a linear exchange economy. However, there is a combinatorial characterization for existence due to Gale [13], and we discuss this characterization and other properties of the linear exchange economy in detail in Section 2.

Observe that pareto efficiency is not a particularly restrictive notion: an allocation is efficient unless there is an alternative that is *universally* agreed to be better (or at least as good). This requirement of unanimity has important implications. Allocations that may

[1] As well as the possibility of non-concave utility function, numerous other factors may affect the practicality of the welfare theorems: market power and the presence of price-makers; incomplete or asymmetric information; externalities; convergence issues for equilibria; the existence of multiple equilibria; economies of scale when production is added to the exchange economy, etc. Such issues are not our focus here.

be viewed as societally better outcomes may be blocked by a single agent. For example, pareto allocations can be extremely inequitable. The second welfare theorem attempts to address this concern: Any pareto solution can be supported as a Walrasian equilibrium. Specifically, by redistributing the initial endowments via lump-sum payments, a set of prices exists that can sustain any pareto solution. (The second theorem also requires concave utility functions.) Thus, the second welfare theorem implies that we can separate out issues of economic efficiency from issues of equity.

As stated, however, the second theorem is of limited practical value due to the infeasibility of direct transfer payments. Thus our goal is to obtain non-redistributive second welfare theorems. Specifically, maximising the social welfare, $\sum_{i=1}^{n} u_i(\mathbf{x}_i)$, is a fundamental question in economics; so, can we support at equilibrium an allocation with high social welfare? For example, in a linear exchange economy it is particularly easy to find an optimal social allocation. For each good $j$, simply give all of it to the trader $i$ for whom it proffers the greatest utility per unit. However, even in this basic case, a Walrasian equilibrium may produce very low social welfare. Intuitively the reason is simple: a trader with a large utility coefficient for a good may not be able to afford many units of it. This may be because (a) the good is in high demand and thus has a high price and/or (b) the trader has a small budget because the goods she initially possesses are abundant and, thus, have a low price.

On the other hand, the second welfare theorem tells us that, with redistribution, it is possible to find prices that support an allocation of optimal welfare. Can any more practical, market-based mechanisms achieve this? To answer this we consider a mechanism that is allowed to cluster the traders into trading groups.

### 1.3 Market Clustering.

Suppose we partition the traders into $k$ separate markets, for some integer $k$. In each market $t$, trade then proceeds as normal with a distinct set of Walrasian prices $\mathbf{p^t}$ generated. The combined effect of this clustering is an allocation that may be very different than would have resulted from a single market. So, first, can market clustering be used to improve social welfare? If so, second, can it be used to optimize social welfare?

The answer to the first question is `yes`. Trivially, the option to segment the market cannot hurt because we could simply place all the traders in the same market anyway. In fact, market clustering may dramatically improve social welfare; we give an example in Section 3 where the ratio between the social welfare with two markets and the social welfare with one market is unbounded.

The answer to the second question, however, is `no`. Not every pareto solution can be supported by market clustering. In particular, there are cases where the optimal social solution cannot be obtained by clustering. Indeed, we give an example in Section 3 where the ratio between the optimal social welfare and the optimal welfare that can be generated by market clustering is also unbounded.

The main focus of this paper then becomes to efficiently obtain as large a welfare as possible under market clustering.

We remark that the basic idea underlying market clustering, i.e. the grouping and separation of traders, is a classical one in both economic theory and practice. In particular, it lies at the heart of the theory of trade. On the one hand, countries should trade together (grouping) to exploit the laws of comparative advantage; on the other hand, trade between countries may be restricted (separation) to protect the interests of certain subsections (e.g. specific industries or classes of worker). Interestingly, of course, whilst separation has a net negative effect on welfare in international trade models, our results show it can have a large net positive effect in general equilibrium models. Other examples that can be viewed as market clustering arise in the regulation of oligopolies and in the issue of trading permits. A less obvious example concerns bandwidth auctions where participants are grouped into "large" (incumbant) and "small" (new-entrants). Trade, with the mechanism in the form of feasible bidding strategies, is then restricted depending upon the group.

### 1.4 Our Results.

**A $k$-*equilibrium* is a partition of the set of traders into $k$ markets together with an equilibrium for each of these $k$ markets.**

Our main result, in Section 6, is a polynomial time algorithm that finds an $\varepsilon$-approximate $k$-equilibrium, of almost optimal social welfare, provided the number of goods and markets are constants. The key to this result is a limit theorem in Section 5 showing that, in a single market, $\varepsilon$-approximate equilibria converge to what we call a *generalized equilibrium.*

On the other hand, in Section 4 we show that it is NP-hard to find an optimal $k$-equilibrium in a linear exchange economy with a bounded number of good and markets. The restriction to a bounded number of goods is necessary to obtain any reasonable approximation guarantees; for linear exchange economies with an unbounded number of goods, the problem is as hard as approximating the maximum independent set problem, even for the case of just 2 markets.

## 2 Walrasian Equilibria in the Linear Exchange Model

Take an equilibrium with prices $\mathbf{p}$ and allocations $\mathbf{x_i}$ for the Walrasian model with linear utilities. Recall, we may assume that the following hold:

`Budget Constraints:` Trader $i$ cannot spend more than she receives: $\mathbf{p} \cdot \mathbf{x_i} \leq \mathbf{p} \cdot \mathbf{e_i}$ (1)

`Optimality:` Each trader $i$ optimizes the bundle of goods she buys: (2)
$\mathbf{u_i} \cdot \mathbf{x_i}$ is maximized subject to (1)

`Market Clearing:` Demand does not exceed supply, for any $g_j$: $\sum_i x_{ij} \leq \sum_i e_{ij}$ (3)

### 2.1 Properties of Equilibria

The following claims are well-known facts (see *e.g.* [16]).

**Claim 1.** *At equilibrium, the budget constraint* (1) *is tight for any trader.*

**Claim 2.** *At equilibrium, the market clearing condition* (3) *is tight for any $g_j$ with $p_j > 0$.*

**Claim 3.** *At equilibrium, for any subset $S$ of traders,*

$$\text{there is a good } g_j \text{ s.t. } \sum_{i \in S} x_{ij} > \sum_{i \in S} e_{ij} \quad \text{iff} \quad \text{there is a good } g_{j'} \text{ s.t. } \sum_{i \notin S} x_{ij'} > \sum_{i \notin S} e_{ij'}$$

**Claim 4.** *At equilibrium, for any $i$ with $P_i := \mathbf{p} \cdot \mathbf{e}_i > 0$ and any $g_j$ with $p_j > 0$*

$$\frac{u_{ij}}{p_j} \leq \frac{\mathbf{u}_i \cdot x_i}{P_i} \tag{4}$$

*Moreover, the inequality is tight for any $i, j$ with $x_{ij} > 0$.*

## 2.2 The Existence of Equilibria in a Single Market

Gale [13] gave a characterisation for when linear exchange economies possess equilibria. Observe that the price of every good will be strictly positive provided each good is owned by at least one trader, and at least one trader desires it. We may assume this is the case as any good that does not satisfy this condition may be removed from the model; in this case supply will exactly equal demand for each good. Gale also assumes every trader is *non-altruistic* in that they each desire at least one good. (We say that a trader $i$ is an *altruist* if $u_{ij} = 0$ for every good $j$.)

**Theorem 5.** *[13] An altruist-free linear exchange economy has a Walrasian equilibrium if and only if there is no super self-sufficient set of traders.* □

Here, a subset $S$ of traders is called *super self-sufficient* if

1. `Self-Sufficiency:` $\sum_{i \notin S} e_{ij} = 0$ for every good $g_j$ such that $\sum_{i \in S} u_{ij} > 0$.
2. `Superfluity:`There is a good $g_j$ such that $\sum_{i \in S} e_{ij} > 0$ and $\sum_{i \in S} u_{ij} = 0$.

It will be useful to reinterpret Gale's condition combinatorially using the market graph. The *market graph* $G_M$ for a given market is a directed graph whose set of vertices is the set of goods in that market. There is an arc $j \to j'$ with label $i$ if there is a trader $i$ with $e_{ij} > 0$ and $u_{ij'} > 0$; thus trader $i$ has good $g_j$ and, depending upon the prices, is willing to trade it for good $g_{j'}$. Let $h(S)$ denote the goods that are the heads of arcs with labels from traders in $S$, and let $t(\bar{S})$ denote the goods that are the tails of arcs corresponding to traders not in $S$. Then $S$ is self-sufficient if $h(S) \cap t(\bar{S}) = \emptyset$. In this case, $h(S)$ induces a directed in-cut in the market graph. (Thus, a sufficient – but not necessary – condition for the existence of an equilibrium is the strong connectivity of the market graph. Moreover, any directed cut will correspond to a self-sufficient set.) If, in addition, $h(S)$ is a strict subset of $t(S)$, then $S$ is super self-sufficient. For example, the market graph shown in Figure 1 does not have an equilibrium. It represents a market with 6 traders and 5 goods $g_1, \dots g_5$: each arc $g \xrightarrow{i} h$ represents one trader $i$ with $e_{ig} = 1$ and $u_{ih} = 1$, all other values being 0. Then traders $\{4, 5, 6\}$ form a super self-sufficient set, so this market does not have an equilibrium.

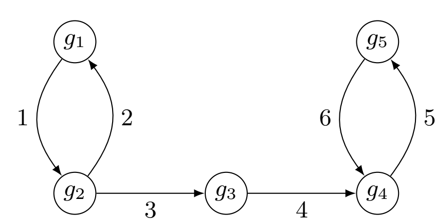


**Fig. 1.** A market graph with a super self-sufficient set and, therefore, no equilibrium.

We can use the market graph to test Gale's condition efficiently. Furthermore, Jain [16] gave a polynomial time algorithm to find an equilibrium provided one does exist.[2]

[2] In the linear exchange model the equilibria need not be unique. However, in a single market all equilibria give the same welfare [13].

### 2.3 The Existence of Equilibria in a Market Clustering.

Recall a trader $i$ is an altruist if $u_{ij} = 0$ for every good $j$. An economy is *altruistic* if it is allowed to contain altruistic traders. It is important for us to understand the implications of altruism even for economies without altruists. This is because clustering may create de facto altruists in the submarkets. Moreover such altruists are one of the factors that allow the equilibria to exist in a market clustering, even if the single market has no equilibrium. We can easily extend Gale's theorem to altruistic economies.

**Theorem 6.** *An altruistic, linear exchange economy has an equilibrium if and only if every super self-sufficient set of traders contains at least one altruist.*

*Proof.* An altruist $i$ is willing to trade with any agent as she is, trivially, equally happy to possess any good. So in the market graph corresponding to an economy with altruistic traders we may include arcs from any good altruist $i$ owns to every other good. Thus, trader $i$ cannot be in any set $S$ for which $h(S)$ forms a directed in-cut. □

So altruistic economies need not have equilibria. However, they can always be clustered into markets with equilibria provided the number of markets $k$ is at least the number of goods $m$.

**Theorem 7.** *An altruistic, linear exchange economy with $m$ goods has an $m$-equilibrium.*

*Proof.* We prove this by induction on the number of goods. An altruistic economy with one good $g_j$ has a trivial equilibrium. Now take an altruistic economy with $m$ goods. If it has no super self-sufficient set of traders consisting entirely of non-altruists then, by Theorem **??**, it has an equilibrium. Otherwise let $S$ be a minimal super self-sufficient set of non-altruists. By minimality, the market induced by $S$ contains an equilibrium as it has no super self-sufficient subset.

As they are not altruistic, each trader in $S$ desires at least one good. By definition, however, traders in $S$ desire no goods held by traders in $\bar{S}$. So there is at least one good held by $S$ that is not held by traders in $\bar{S}$. Thus the market induced by the traders in $\bar{S}$ contains at most $m-1$ goods. By induction it can be partitioned into $m-1$ clusters that each have an equilibrium. Together with the cluster $S$, we obtain an $m$-equilibrium. □

For example, consider again the market in Figure 1. If we partition the traders into two, with trader 3 alone in the first market and traders $\{1, 2, 4, 5, 6\}$ in the second market, then both resulting markets have an equilibrium (with $x_{32} = 1$ in the first market, and $p_3 = 0$ in the second market).

Theorem 7 is also tight. There are examples where $m$ clusters are needed otherwise at least one of them has no equilibrium. For example, take a market with $m$ traders. Trader $i$ is interested only in good $g_i$, and has a positive endowment of goods $g_i, g_{i+1}, \ldots, g_m$. It is easy to check that any cluster that contains more than one trader contains a super self-sufficient set.

## 3 Single Markets, Market Clustering and Welfare Redistribution.

In this section, we examine the potential benefits of market clustering and the limits of its power as a tool. First, we have seen that equilibria may not exist in the single market case (i.e. when market clustering is prohibited). In such instances, by Theorem 7, market clustering can always be applied to produce equilibria. Furthermore, even when equilibria do exist in the single market case, market clustering may lead to huge improvements in social welfare in comparison. On the other hand, market clustering is not as powerful as welfare redistribution; specifically, market clustering does not always support every pareto allocation. To see this we consider two measures regarding the social welfare function:

1. `The Clustering Ratio:` the ratio between the maximum social welfare under market clustering and the social welfare obtained in a single market.
2. `The Redistribution Ratio:` the ratio between the maximum achievable social welfare (with welfare redistribution) and the maximum welfare under market clustering.

In Appendix A we give examples to show that both ratios can be unbounded. Indeed, the clustering ratio can be unbounded even if we are restricted to two clusters, and the redistribution ratio can be unbounded even if we can partition into an unlimited number of clusters.

## 4 The Hardness of Market Clustering

In this section, we consider the hardness of the $k$-market clustering problem. First we show in Appendix B that the problem is NP-hard even if we only have a fixed number of goods and a fixed number of markets, that is, $m$ and $k$ are constant.

**Theorem 8.** *Given an instance of the* 2*-market clustering problem with five goods and linear utility functions, it is NP-hard to decide whether there is a clustering that yields a social welfare of value at least $Z$, for any $Z > 0$.*

(We provide a partial complement to this result by giving in Section 6 a polynomial time algorithm to find an approximate Walrasian equilibria when there are a fixed number of goods and a fixed number of markets).

Then we show, in Appendix B, a much stronger hardness result for the market clustering problem when the number of goods is unbounded.

**Theorem 9.** *For any constant $\delta > 0$ and maximum social welfare $Z$, unless* NP = ZPP, *it is hard to distinguish between the following two cases:*

- `Yes-Instance:` *There is a clustering that yields a social welfare of value at least $Z^{1-\delta}$.*
- `No-Instance:` *There is no clustering that yields a social welfare of value at least $Z^{\delta}$.*

## 5 Approximate Walrasian Equilibria.

We are interested in finding an *$\varepsilon$-approximate market equilibrium*; that is, for each market, our algorithm outputs a price $\mathbf{p}$ and an allocation $\mathbf{x}$ satisfying the following conditions.

- `Budget Constraints:` Trader $i$ cannot spend more than she receives: $\mathbf{x}_i \cdot \mathbf{p} \leq \mathbf{e}_i \cdot \mathbf{p}$
- `Approximate Optimality:` Subject to the budget constraints, each trader $i$ purchases a bundle $\mathbf{x}_i$ whose utility is similar to that of the optimal bundle $\mathbf{x}_i^*$: $\mathbf{u}_i \cdot \mathbf{x}_i \geq (1-\varepsilon)\mathbf{u}_i \cdot \mathbf{x}_i^*$
- `Market Clearing:` Demand never exceeds supply: for any $g_j$, $\sum_i x_{ij} \leq \sum_i e_{ij}$

### 5.1 Existence of Approximate Walrasian Equilibria.

Compared to exact market equilibria, which do not always exist, there is always an approximate market equilibrium with arbitrary small approximation:

**Theorem 10.** *For $\varepsilon > 0$, every market has a market-clearing $\varepsilon$-approximate equilibrium.*

By *market-clearing $\varepsilon$-approximate equilibrium*, we mean an $\varepsilon$-approximate equilibrium for which the approximate market clearing inequality is tight. This theorem can be inferred from the algorithm of [15]. For completeness, we give a direct proof of this fact in Appendix C.

### 5.2 Properties of Approximate Walrasian Equilibria.

We now discuss some properties of equilibria that will later be very useful to us in designing efficient algorithms. Given a market, we use the following definitions:

- $u_{\max} = \max_{i,j} u_{ij}$ is the maximum coefficient of any utility function.
- $u_{\min} = \min_{i,j:u_{ij}>0} u_{ij}$ is the minimum *non-zero* coefficient of any utility function.
- $p_{\max} = \max_j p_j$ is the maximum price of any good in the market.
- $p_{\min} = \min_{j:p_j>0} p_j$ the minimum *non-zero* price of any good in the market.
- $e_{\min} = \min_{i,j:e_{ij}>0} e_{ij}$.

Assume wlog that $\sum_i e_{ij} = 1$, for every good $g_j$. We can connect the above values via the market graph. Recall that $n$ denotes the number of traders and $m$ the number of goods; then we obtain

**Lemma 1.** *If the market graph is strongly connected then, at a market equilibrium,*

$$\frac{p_{\max}}{p_{\min}} \leq e^{\frac{nm}{e}} \left( \frac{u_{\max}}{e_{\min} u_{\min}} \right)^{nm}$$

*In particular, scaling so that $p_{\min} = 1$ gives $p_{\max} \leq e^{nm/e} \left( \frac{u_{\max}}{e_{\min} u_{\min}} \right)^{nm}$.*

*Proof.* We may assume (solely for the duration of this proof) that each trader has a positive endowment for exactly one good and no two traders have positive endowments for the same good [16]. To do this, consider a trader $i$ with endowment $\mathbf{e_i}$. For each good $g_j$ such that $e_{i,j} > 0$ we create a new trader $i_j$ with $e_{i_j j} = e_{ij}$, $\mathbf{u}_{i_j} = \mathbf{u}_i$ and $e_{i_j j'} = 0$, for all $j' \neq j$. So each trader now has only one good. Furthermore, if two traders have the same good, then we simply give the good two different names (and replicate the utility functions of other traders accordingly). Now, each trader represents a unique good, i.e., a trader $i$ has a positive endowment for the unique good $g_i$. So we have at most $nm$ traders/goods.

These transformations maintain the strong connectivity of the market graph. Moreover, all the copies of the same original good will have the same price in an equilibrium. After these transformations, the number of units of each good will in general be less than one. Thus, we scale all the initial endowments so that each trader $i$ has one unit of a good $g_i$. In addition, we must scale the coefficients of utility functions; otherwise, the scaling would effect the social welfare. Specifically, for each good $i$, we divide the initial endowments of trader $i$ by $e_{ii}$, and we multiply the utilities of every trader for this good by $e_{ii}$, so as to keep the prices unchanged.

We may assume that no good has a price of zero. By Equation (4), we have:

$$\frac{u'_{ij}}{p_j} \leq \frac{\sum_\ell u'_{i\ell} \cdot x_{i\ell}}{p_i}$$

For a pair $i, j$ with $u'_{ij} > 0$, we get:

$$\frac{p_i}{p_j} \leq \frac{\sum_\ell u'_{i\ell} \cdot x_{i\ell}}{u'_{ij}} \leq \frac{u'_{\max}}{u'_{\min}} \sum_\ell x_{il}$$

Assume $p'_{\min} = p'_{i_0}$ and $p'_{\max} = p'_{i_s}$, where $s \leq nm$. Because the market is strongly connected, there is a sequence of traders with indices $i_0, i_1 \ldots, i_s$, $s \leq nm$, such that $u'_{i_{j-1}i_j} > 0$ for all $i \in \{1, \ldots, s\}$. Multiplying the previous inequalities for all consecutive terms of this sequence, we get:

$$\frac{p_{\max}}{p_{\min}} = \prod_{j=0}^{s-1} \frac{p_{i_{j+1}}}{p_{i_j}} \leq \left(\frac{u'_{\max}}{u'_{\min}}\right)^s \cdot \prod_{j=0}^{s} \left(\sum_\ell x_{j\ell}\right) \leq \left(\frac{nm}{s}\right)^s \left(\frac{u'_{\max}}{u'_{\min}}\right)^{nm} \leq e^{\frac{nm}{e}} \left(\frac{u'_{\max}}{u'_{\min}}\right)^{nm}$$

Here, the second inequality follows from the Arithmetic-Geometric Mean Inequality and the fact $\sum_{j=0}^{s} \sum_\ell x_{j\ell} \leq \sum_{j=0}^{s} \sum_\ell e_{j\ell} \leq nm$ by the market clearing constraint (3). Now, observe that there is a pair $i, j$ such that $u'_{\min} = e_{jj} u_{ij} \geq e_{\min} u_{\min}$, and there is a (different) pair $i, j$ such that $u'_{\max} = e_{jj} u_{ij} \leq u_{\max}$. □

The same reasoning applied to approximate equilibria gives:

**Lemma 2.** *If the market graph is strongly connected, at an $\varepsilon$-approximate market equilibrium:*

$$\frac{p_{\max}}{p_{\min}} \leq e^{nm/e} \left(\frac{u_{\max}}{(1-\varepsilon) u_{\min} e_{\min}}\right)^{nm}$$

It is possible for a market that is not strongly connected to have an equilibrium: the owners of the goods reachable from a strongly connected set induce a self-sufficient set but not necessarily a super self-sufficient set. However, in this case we cannot bound the ratio $p_{\max}/p_{\min}$ as seen from the following lemma.

**Lemma 3.** *Consider a market with equilibrium $\mathbf{p}, \mathbf{x}$. Let $W$ be a proper subset of goods such that for any player $i$, if there is some $j \in W$ with $e_{ij} > 0$, then $u_{ik} = 0$ for all $k \notin W$ (i.e. $W$ is the shore of a directed cut in the market graph). Then, for any $B > 1$, $\mathbf{p}', \mathbf{x}$ is also an equilibrium where:*

$$p'_j = \begin{cases} p_j & \text{if } j \notin W, \\ Bp_j & \text{if } j \in W. \end{cases}$$

*Proof.* The lemma follows from the following two facts. First, $x_{ij} = 0$ for any good $g_j \notin W$ and any trader $i$ with $\sum_{k \in W} e_{ik} > 0$ since then $u_{ij} = 0$ by the definition of $W$. Second, $x_{ij} = 0$ for any trader $i$ with $\sum_{k \in W} e_{ik} = 0$ and $g_j \in W$. Consequently, scaling the prices of goods in $W$ does not effect the equilibrium. □

An implication of this lemma is that the strongly connected components have price allocations that are essentially independent of each other: for example one could decompose the problem, find local equilibria in each component, and then scale the prices accordingly to get a global equilibrium. Also, again by scaling the prices of $W$, we can assume that the minimum price in $W$ is no more than $u_{\max}/u_{\min}$ times the maximum price outside

$W$, as it does not change the optimality of the allocations (it would be a problem if there was a trader with $\sum_{j \notin W} u_{ij} = 0$ and $\sum_{j \notin W} e_{ij} = 0$, but then taking this trader plus $\{i : \sum_{j \in W} e_{ij} > 0\}$ would give a super self-sufficient set). This gives the following strengthening:

**Lemma 4.** *Any market having an equilibrium has one such that:*

$$\frac{p_{\max}}{p_{\min}} \leq e^{\frac{nm}{e}} \left( \frac{u_{\max}}{e_{\min} u_{\min}} \right)^{nm}$$

### 5.3 Limits of Equilibria

By Theorem 10, for any $\varepsilon > 0$ there is a market-clearing $\varepsilon$-approximate equilibrium. When $\varepsilon$ tends to 0, the prices of these approximate equilibria may diverge (if no exact equilibrium exists), but the allocations of goods to players, as they are chosen from a compact set, admit at least one limit point, an allocation $\mathring{\mathbf{x}}$. We call such an allocation a *limit allocation*. In particular, if the market admits an exact equilibrium, then $\mathring{\mathbf{x}}$ is the allocation of an exact equilibrium (if $\mathring{\mathbf{x}}$ could not be obtained as a limit of approximate equilibria with converging prices, one could exhibit a super self-sufficient set and this would be a contradiction). In any case, $\mathring{\mathbf{x}}$ satisfies the market clearing constraints with equality.

The allocation $\mathring{\mathbf{x}}$ may not be supported by a set of real prices. For example, there is obviously no set of prices supporting $\mathring{\mathbf{x}}$ when the market does not have an exact equilibrium. We show that $\mathring{\mathbf{x}}$ can be supported by taking prices from a more general set than the real numbers. Consider the set $Q = \mathbb{N} \times \mathbb{R}_+$, our new set of "prices". We denote by $\pi_1$ and $\pi_2$ the first and second projection, *i.e.* $\pi_1(x, y) = \pi_2(y, x) = x$. We extend these projections to vectors (and abuse notation) by: $\pi_i\left((v_j)_j\right) = \left(\pi_i(v_j)\right)_j$. We then redefine the notion of equilibrium in terms of $Q$. For $p \in Q^m$ and $\mathring{\mathbf{x}} \in \mathbb{R}_+^{m \times n}$, let the *rank* $r_i$ of $i$ be the maximum $a$ such that $\sum_{j\,:\,\pi_1(p_j)=a} e_{ij} > 0$, for all $i$. The pair $\mathbf{p}, \mathbf{x}$ is a *generalized equilibrium* if:

– `Budget Constraints:` For all $i \in \{1, \ldots, n\}$, for all $a \geq r_i$,

$$\sum_{j\,:\,\pi_1(p_j)=a} \pi_2(p_j) \cdot \mathbf{x}_{ij} \leq \sum_{j\,:\,\pi_1(p_j)=a} \pi_2(p_j) \cdot \mathbf{e}_{ij}$$

– `Optimality:` For each trader, $\mathbf{x}_i$ maximizes the utility $\mathbf{u}_i \cdot \mathbf{x}_i$ over all allocations satisfying the budget constraint.
– `Market Clearing:` No good is in deficit:

$$\sum_i x_{ij} \leq \sum_i e_{ij} \qquad \text{for all goods } j \text{ with } \pi_2(p_j) > 0.$$

This is indeed a generalization: if we force the prices to be in $\{0\} \times \mathbb{R}_+$ then a generalized equilibrium would give a Walrasian equilibrium. An *$\varepsilon$-approximate generalized equilibrium* is defined by replacing the optimality condition by: $\mathbf{u}_i \cdot \mathbf{x}_i$ is at least $(1 - \varepsilon)$ times the utility of a best response of trader $i$, for any trader $i$.

**Theorem 11.** *For any market, each limit allocation $\mathring{\mathbf{x}}$ gives a generalized equilibrium.*

The proof of Theorem 11 is given in Appendix D. A generalized equilibrium can be approximated by an approximate Walrasian equilibrium with almost as high welfare. The converse is not true, an approximate equilibrium may achieve a welfare arbitrarily high compared to a generalized equilibrium; consider the market with two traders and two goods, $e_{11} = e_{22} = 1$, $u_{12} = L$, $u_{22} = 1$, all the other values are zero. In this market, the only generalized equilibrium has welfare 1, but there are approximate equilibrium with welfare $\varepsilon \cdot L + (1 - \varepsilon)$, and this tends to $+\infty$ when $L$ tends to $+\infty$.

**Lemma 5.** *Let $\mathring{\mathbf{x}}, \mathring{\mathbf{p}}$ be a generalized equilibrium. For any $\varepsilon > 0$, there is an approximate equilibrium with total welfare at least $1 - \varepsilon$ times the welfare of $\mathring{\mathbf{x}}, \mathring{\mathbf{p}}$.*

*Proof.* (Sketch: the proof is similar to the proof of Theorem 10.) Fix $M$ such that $u_{ij}/(M \cdot \pi_2(\mathring{p}_j)) > u_{ij'}/\pi_2(\mathring{p}'_j)$ for any $i$, $j$ and $j'$ with $\mathring{x}_{ij'} \neq 0$ and $\pi_1(\mathring{p}_j) > r_i$, and $\sum_{r_i < a} e_{ij}\pi_2(\mathring{p}_j) < \varepsilon \cdot M \cdot \max_{i:r_i > a} \sum_{j:u_{ij} \neq 0} \pi_2(\mathring{p}_j) x_{ij}$. Set $p_j = M^{\pi_1(\mathring{p}_j)}\pi_2(\mathring{p}_j)$.

The first set of constraints on $M$ state that every trader achieving a positive welfare in $\mathring{\mathbf{x}}, \mathring{\mathbf{p}}$ is at a best response in $\mathring{\mathbf{x}}, \mathbf{p}$. Then we set $\mathbf{x}' = (1 - \varepsilon)\mathbf{x}$. Then, to each trader $i$ with welfare 0 in $\mathring{\mathbf{x}}, \mathring{\mathbf{p}}$, we allocate to $i$ a best response $\mathbf{x}'_i$. Because of the second set of constraints on $M$, these traders don't claim more than $\varepsilon$ of what was allocated to any other trader in $\mathring{\mathbf{x}}$. Finally, we redistribute the goods in excess in $\mathbf{x}'$ to traders with excessive budget, to get a market-clearing $\varepsilon$-approximate equilibrium $\mathbf{x}, \mathbf{p}$.

## 6 A Fully Polynomial Time Approximation Scheme

In this section, we exploit the structure we have now developed to obtain a polynomial time algorithm to find an $\varepsilon$-approximate equilibrium for the $k$-market clustering problem where the number of goods $m$ and the number of markets $k$ are constant. Moreover, this equilibrium has a very strong welfare guarantee: it gives social welfare of at least $1 - \varepsilon$ times the welfare of the optimal $k$-cluster generalised equilibrium.

**Theorem 12.** *For any $\varepsilon > 0$ and for fixed $k, n \in \mathbb{N}$, there is an algorithm that, given a market $M$ with $m$ goods, computes within time polynomial in $\frac{1}{\varepsilon}$ and the size of the $M$, an $\varepsilon$-approximate generalized $k$-equilibrium for $M$ with welfare at least $1 - \varepsilon$ the optimal welfare of a generalized $k$-equilibrium.*

### 6.1 The Dynamic Program

Our dynamic program will take as an input a set of generalized prices for every good in each market. This follows as we may try all possible prices selected from a finite set of prices in $\{1, \ldots, m\} \times P$ where $P = \{1, 1 + 1/b, (1 + 1/b)^2, \ldots, (1 + 1/b)^{\sigma - 2}\}$. Here $b \in \mathbb{N}_+$ is a parameter to be set later, and $\sigma$ is such that $(1 + 1/b)^{\sigma - 3} \leq e^{nm/e} \left(\frac{m \cdot u_{max}}{e_{min} u_{min}}\right)^{nm} \leq (1 + 1/b)^{\sigma - 2}$. A generalized price $(a, p)$ encodes a real price $\nu(a, p)$ given by $M^a \cdot p$ where $M$ is an arbitrarily large constant. $M$ is only used algebraically, we won't need to set its value. So, we have $m \cdot \sigma$ possible prices for each good in each market. Thus, the number of combinations of prices to is $(m\sigma)^{km}$, where $k$ is the number of markets and $m$ is the number of goods.

Given the estimated prices, the dynamic program runs over each market to compute an approximate equilibrium that maximizes the total social welfare. We denote the estimated prices in market $t$ by $\mathbf{p}^t \in (\{1, \ldots, m\} \times P)^m$, *i.e.* $p^t_j$ is the price of a good $j$ in a market $t$. We denote an initial endowment and a final allocation in a clustered market by $\mathbf{e}^t_i$ and $\mathbf{x}^t_i$. (Thus, $\sum_{t=1}^k \mathbf{e}^t_i = \mathbf{e}_i$.)

The algorithm considers each trader iteratively. At each iteration, it assigns the trader to a market and gives a near-optimal bundle to this trader, according to her utility function and the prices in that market. Once the $i$th trader is assigned, the algorithm only remembers the deficit (or surplus) of each good in every market – this will be key in obtaining an efficient algorithm. Once every trader is assigned, it selects the best possible solution that satisfies the approximate market clearing constraint, *i.e.* the deficits must be small. Thus, we encode the state of each market $t$ by a vector $\mathbf{y}^t$, where $y^t_j$ denotes the surplus (or deficit) of the good $j$. Let $I_t$ denote the set of traders already in the market $t$, and $\mathbf{x}_i, i \in I_t$ the bundles given to these traders. Ideally, we would like to have $y^t_j = \sum_{i\in I_t} e_{ij} - \sum_{i \in I_t} x_{ij}$. Hence, the value of $y^t_j$ could be any real value between $-1$ and 1. However, as we cannot afford to store all possible values for $y^t$, we round these values into a set $\widetilde{W}$ of cardinality $\alpha \cdot 4n$, where $\alpha$ will be set later. To define $\widetilde{W}$ we first define a coarser set $W$.

$$W = \left\{ \left(\frac{b}{b+1}\right)^{\alpha}, \left(\frac{b}{b+1}\right)^{\alpha-1}, \dots, \frac{b}{b+1}, 1 \right\}.$$

We then choose $\alpha$ to be minimal such that

$$\left(\frac{b}{b+1}\right)^{\alpha} \le \frac{e_{\min}}{2n(b+1)} \quad \text{and} \quad \left(\frac{b}{b+1}\right)^{\alpha} < \frac{u_{\min} \cdot e_{\min}}{m \cdot u_{\max} \cdot p_{\max}} \cdot \frac{1}{(b+1)^2}$$

Observe that $W$ induces a set of intervals $\left[(b/(b+1))^{\ell}, (b/(b+1))^{\ell-1}\right]$, for $\ell = 1, \dots \alpha$. We can now create the set $\widetilde{W}$ by dividing each interval of $W$ and its negation into subintervals. Specifically, for each interval $\left[(b/(b+1))^{\ell}, (b/(b+1))^{\ell-1}\right]$ (resp., $\left[-(b/(b+1))^{\ell-1}, -(b/(b+1))^{\ell}\right]$), we divide $W$ equally into $2n(b+1)$ subintervals and put the boundary points in $\widetilde{W}$. Thus,

$$\widetilde{W} = \bigcup_{\ell=1}^{\alpha-1} \left\{ \left(\frac{b}{b+1}\right)^{l} \left(1 + \frac{q}{2nb(b+1)}\right) : q \in \{0, 1, \dots, 2n(b+1)\} \right\} \cup \{0\} \cup$$
$$\bigcup_{\ell=1}^{\alpha-1} \left\{ \left(-\frac{b}{b+1}\right)^{l} \left(1 + \frac{q}{2nb(b+1)}\right) : q \in \{0, 1, \dots, 2n(b+1)\} \right\}$$

We insist that the algorithm selects allocations of goods that take values from $W$, and we then round down the surplus (or deficits) to values in $\widetilde{W}$. Formally, $\mathbf{x}^t \subseteq W^{n \times m}$ and $\mathbf{y}^t \subseteq \widetilde{W}^m$ for any market $t$. Towards this goal, let $\lfloor a \rfloor_{\widetilde{W}}$ denote the value of $a$ rounded down to the closest value in $\widetilde{W}$, i.e., $a' = \lfloor a \rfloor_{\widetilde{W}}$ is the largest value in $\widetilde{W}$ such that $a' \le a$.

We now need to ensure that these allocations correspond to an approximate generalized equilibrium (which in turn will correspond to an approximate equilibrium). To do this, for a market $t$ and a trader $i$, we say that an allocation $x_i$ of goods is *compatible* with $i$ and $t$ if, for $r_i = \max_{j:e_{ij}>0} \pi_1(p_j)$:

- if $u_{ij} > 0$, then $\pi_1(p^t_j) \ge r_i$,
- if $x_{ij} > 0$, then $\pi_1(p^t_j) \le r_i$,
- $\sum_{j:\pi_1(p^t_j) \ge r_i} \pi_2(p^t_j) \cdot x_{ij} \le \sum_{j:\pi_1(p^t_j)\ge r_i} \pi_2(p^t_j) \cdot e_{ij}$ (Budget constraint).

An allocation $\mathbf{x}_i$ compatible with $i$ and $t$ is *nearly-optimal* if $\mathbf{u}_i \cdot \mathbf{x}_i \ge (1-\varepsilon) \max_z \mathbf{u}_i \cdot \mathbf{z}$ where $\mathbf{z}$ ranges over all the allocations compatible with $i$ and $t$. By this definition, assuming $M$ is large enough, a nearly-optimal allocation is an approximate best response for trader $i$ in market $t$. The recurrence relation of our dynamic programming algorithm is then

$$f(0, \mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^k) = \begin{cases} 0 & \text{if } \mathbf{y}^1 = \mathbf{y}^2 = \dots = \mathbf{y}^k = \mathbf{0} \\ -\infty & \text{otherwise} \end{cases}$$

$$f(i, \mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^k) = \max_{t \in \{1,2,\dots,k\}, \mathbf{x}_i} f(i-1, \mathbf{y}^1, \dots, \lfloor \mathbf{y}^t - \mathbf{e}_i + \mathbf{x}_i \rfloor_{\widetilde{W}}, \dots, \mathbf{y}^k) + \mathbf{u}_i \mathbf{x}_i$$

where $\mathbf{x}_i$ ranges over the nearly-optimal allocations compatible with $i$ and $t$.

Finally, an allocation is *valid* if $0 \leq \mathbf{y}^t \leq 1 - (b/(b+1))^3$ for all $t$, i.e., there is no market having a positive deficit, or a surplus greater than $1 - (b/(b+1))^3$. Notice that some of the surplus may have been lost in the rounding steps; we will show later that these additional losses amount to a fraction at most $1/(b+1)$ of each good. Hence, if $f(n, \mathbf{y}^1, \ldots, \mathbf{y}^t)$ is finite for some valid allocation, and provided that $b$ is sufficiently large, it gives an approximate *generalized* equilibrium: the budget constraints and approximate optimality constraints are guaranteed by the restriction on the choice of $\mathbf{x}_i$ at each step, and market clearing is guaranteed by the validity of the allocation.

This completes the description of the dynamic program. It remains to compare the total social welfare to that of the optimal clustering and to analyse the running time of the algorithm.

### 6.2 Quality Analysis

Because of the rounding step, our dynamic programming algorithm loses some fraction of each good $j$. We have to bound the number of units of the good $j$ that we lose. By the scaling on $\widetilde{W}$, each time we round $y_j^t$, we lose at most $(2n(b+1))^{-1} \sum_{i \in I_t} e_{ij}$ units when $y_j^t$ is rounded to a positive value, and we lose at most $(b/(b+1))^{\alpha} \leq e_{min}\,(2n(b+1))^{-1}$ units when $y_j^t$ is rounded to zero. By the definition of $e_{min}$, we have $e_{min} \leq \sum_{i \in I_t} e_{ij}$ for all goods $j$. Furthermore, we round down $y_j^t$ at most $n$ times, once for each trader. Thus, summing them up, we lose at most $1/(2(b+1)) \cdot \sum_{i \in I_t} e_{ij} < 1/(2b+2)$ units of each good $j$.

Now, we compare the social welfare of the approximate generalized equilibrium obtained by our algorithm with that of some generalized exact equilibrium.

**Lemma 6.** *For any market $M = (n, m, \mathbf{u}, \mathbf{e})$ and any $\varepsilon > 0$, there is an exact generalized equilibrium such that*

$$\frac{\max_j \pi_2(p_j)}{\min_{j:\pi_2(p_j)\neq 0} \pi_2(p_j)} \leq e^{nm/e} \left( \frac{p_{\max} \cdot m}{e_{\min} \cdot p_{\min}} \right)^{nm}$$

*Proof.* Take an exact generalized equilibrium $\mathbf{x}, \mathbf{p}$ that maximizes the total welfare. Let $G_r := \{j \ : \ \pi_1(p_j) = r\}$, $I_r := \{i \ : \ r_i = r\}$ and $A_r := \{i \in I_r \ : \ \forall j \in G_r, u_{ij} = 0\}$. For each possible rank $r$, let $M_R$ be the market obtained by restricting $M$ to the goods $G_r$. Define $x_{ij}^r = x_{ij}$ and $p_j^r = \pi_2(p_j)$ for any trader $i$ and good $j$ in $M_R$. Fix some rank $r$.

**Claim 13.** *We may assume that $\sum_{j \in G_r} p_j^r x_{ij} = \sum_{j \in G_r} p_j^r e_{ij}$ for any $i$.*

*Proof.* By market-clearing for $\mathbf{x}, \mathbf{p}$, we have $\sum_i x_{ij} = \sum_i e_{ij}$ for any $j \in G_r$. By optimality, we also have $\sum_{j \in G_r} p_j^r e_{ij} = \sum_{j \in G_r} p_j^r x_{ij}$ for any $i \in I_r - A_r$. Substracting the two equalities, we get $\sum_{j \in G_r} \left( \sum_{i \in A_r} p_j^r e_{ij} + \sum_{i: r_i > r} p_j^r e_{ij} \right) = \sum_{j \in G_r} \left( \sum_{i \in A_r} p_j^r x_{ij} + \sum_{i: r_i > r} p_j^r x_{ij} \right)$. Because $u_{ij}^r = 0$ for any $i$ with $r_i > r$ or any $i \in A_r$, we can redistribute the goods of $G_r$ allocated to these traders to fulfill the condition of the claim. □

Then $\mathbf{x}^r, \mathbf{p}^r$ is an equilibrium in $M_r$ because of the previous claim (the optimality constraints and market-clearing constraints follow from the definition of generalized equilibrium).

Let $r$ be some rank and consider $M_r$. We want to compute an approximate equilibrium for $M_r$ such that we can bound the prices. If there is no altruist in $M_r$, apply Lemma **??**.

Otherwise, for any altruist $i$, let $j_i$ such that $p_{j_i} x^r_{ij_i}$ is maximised. Define the utility vector $\mathbf{u}^r$

$$u^r_{ij} = \begin{cases} u_{ij} & \text{if } i \text{ is not an altruist in } M_r, \\ 1 & \text{if } i \text{ is altruist in } M_r \text{ and } j = j_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then in the market defined by $\mathbf{e}$ and $\mathbf{u}^r$, $\mathbf{x}^r, \mathbf{p}^r$ is an approximate solution, where only the altruists of $M_r$ do not follow a best response, and the approximation ratio is at most $(m-1)/m$ by the choice of $\mathbf{u}^r_i$ for altruist trader $i$ (and because there are at most $m$ goods in $M_R$). Hence, by applying Lemma 2 we get $\frac{p_{\max}}{p_{\min}} \leq e^{nm/e} \left(\frac{m u_{\max}}{u_{\min} e_{\min}}\right)^{nm}$ which conludes the proof of the lemma. □

Fix any optimal clustering and consider any market $t$. Take a generalized equilibrium $\mathbf{p}^*, \mathbf{x}^*$ as in Lemma 6. We show that the dynamic program outputs an approximate generalized equilibria with total welfare $(1-\varepsilon)$ times the welfare of $\mathbf{p}^*, \mathbf{x}^*$. This is done by building from $\mathbf{p}^*, \mathbf{x}^*$ a set of generalized prices $\mathbf{p}'$ and allocations $\mathbf{x}'$ computable by the dynamic program.

We assume wlog that $p^*_{min} := \min_{j: p^*_j > 0} \pi_2(p^*_j) = 1$. Denote $p_{\max} := \max_{j: p^*_j > 0} \pi_2(p^*_j)$. $\mathbf{p}'$ is obtained by rounding down the second components of prices to values in $P$, hence $b/(b+1) \cdot \pi_2(\mathbf{p}^*) \leq \pi_2(\mathbf{p}') \leq \pi_2(\mathbf{p}^*)$. Consequently we have for each trader $i$ and $a \geq r_i$:

$$\frac{b}{b+1} \sum_{j: \pi_1(p'_j) = a} \pi_2(p'_j) x^*_{ij} \leq \frac{b}{b+1} \sum_{j: \pi_1(p'_j) = a} \pi_2(p^*_j) x^*_{ij} \leq \frac{b}{b+1} \sum_{j: \pi_1(p'_j) = a} \pi_2(p^*_j) e_{ij} \leq \sum_{j: \pi_1(p'_j) = a} \pi_2(p'_j) e_{ij}$$

Hence, $b/(b+1) \cdot \mathbf{x}^*$ satisfies the budget constraint for the prices $\mathbf{p}'$.

Next, we have to modify $b/(b+1) \cdot \mathbf{x}^*$ further so that it satisfies the condition in our dynamic programming algorithm. Namely, we round down the coefficients of $b/(b+1) \cdot \mathbf{x}^*$ to $W$. This gives an allocation $\mathbf{x}'$ with the properties:

$$\left(\frac{b}{b+1}\right)^2 x^*_{ij} \leq x'_{ij} \leq x^*_{ij} \quad \text{ if } x^*_{ij} \geq \left(\frac{b}{b+1}\right)^{\alpha}, \text{ and } x'_{ij} = 0 \text{ otherwise.}$$

Clearly, $\mathbf{x}'$ also satisfies the budget constraint inequalities. We must check that $\mathbf{x}'_i$ is an almost optimal choice for trader $i$. We get for any good $j$:

$$\left(\frac{b}{b+1}\right)^3 \pi_2(p^*_j) \cdot x^*_{ij} \leq \frac{b}{b+1} \pi_2(p^*_j) \cdot x'_{ij} \leq \pi_2(p'_j) \cdot x'_{ij} \leq \pi_2(p^*_j) x^*_{ij}$$

when $x^*_{ij} \geq \left(\frac{b}{b+1}\right)$ or $x^*_{ij} = 0$, otherwise $u_{ij} x'_{ij} = 0$ and $u_{ij} x^*_{ij} \leq \left(\frac{b}{b+1}\right)^{\alpha} u_{ij}$. We take care of this special case, when $x'_{ij} = 0 < x^*_{ij}$. To this purpose, notice first that the welfare of a trader $i$ with $\sum_{j: \pi_1(p^*_j) = r_i} u_{ij} > 0$ is lower bounded by $\frac{e_{\min} \cdot u_{\min}}{p_{\max}}$, as $e_{\min}$ is the minimum possible budget for a trader (other traders have welfare 0). The maximum ratio utility per unit of price achievable is $\frac{u_{\max}}{p_{\min}}$. Hence, a quantity smaller than $(b/(b+1))^{\alpha}$ of some good bought by a trader contributes to a fraction of her welfare at most $\left(\frac{b}{b+1}\right)^{\alpha} \frac{u_{\max} \cdot p_{\max}}{u_{\min} \cdot e_{\min}}$ which is less than $(1/m)(1/(b+1))^2 \leq (1/m)\left(1 - (b/(b+1))^3\right)$ by the choice of $\alpha$. Hence, over all goods, the fraction of welfare lost in rounding down $\mathbf{x}^*$ is at most $1 - 2(b/(b+1))^3$. This bounds our approximation ratio: $1 - \varepsilon \leq 2\left(\frac{b}{b+1}\right)^3$ which is true for $b = \lceil 3/\varepsilon \rceil$.

With this choice, $\mathbf{x}'$ and $\mathbf{p}'$ satisfy the approximate market equilibrium constraints, thus the dynamic algorithm will find a solution with welfare at least $\mathbf{u} \cdot \mathbf{x}' \geq (b/(b+1))^2 \mathbf{u} \cdot \mathbf{x}^*$

### 6.3 Running Time Analysis.

Now consider the complexity of the dynamic program. We have that:

- $|W| = \alpha + 1$ (by definition),
- $|\widetilde{W}| = 2(\alpha - 1)(2n(b+1) - 1) + 3 = \Theta(\alpha n b)$,
- Complexity for one iteration: $O(k \cdot |W|^m) = O(k\alpha^m)$,
- Number of iterations (for one set of prices): $n \cdot |\widetilde{W}|^{km} = O(n(\alpha n b)^{km})$,
- Total complexity for one set of prices: $O(nk\alpha^m(\alpha n b)^{km})$,
- Number of possible price allocations: $(m\sigma)^{km}$,
- Total complexity: $O(n^{km+1}k\alpha^{(k+1)m}(mb\sigma)^{km})$.

Thus, we have a $(1-\varepsilon)$-approximation algorithm with a running time of

$$O\left(n^{(3k+1)m+1}m^{(2k+1)m}k\left(1+\frac{3}{\varepsilon}\right)^{3km+m}\left(\frac{1}{e}+\log\frac{m\cdot u_{\max}}{e_{\min}u_{\min}}\right)^{(k+1)m}\left(\log\frac{m\cdot u_{\max}}{u_{\min}e_{\min}}\right)^{km}\right)$$

The input of the $k$-market clustering problem in a standard binary representation has a size of $\Omega(n(\log(1/e_{min}) + \log(u_{max}/u_{min}))$. Thus, the running time of our algorithm is polynomial in the size of the input when $m$ and $k$ are constant.

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## Appendix A: Single Markets, Market Clustering and Welfare Redistribution

### The Clustering Ratio.

First, we give a simple example where the clustering ratio is unbounded even if we are constrained to cluster into at most two markets. That is, let $f_1$ be the social welfare at equilibrium with one market and let $f_2$ be the social welfare given by the optimal clustering of the traders into two markets. Then we will show that the ratio $\rho = f_2/f_1$ is unbounded. Consider the following scenario, with three traders and two goods, and $L \gg 1$.

| | | $\mathbf{e}_i$ | | $\mathbf{u}_i$ | |
|---|---|---|---|---|---|
| | | $g_1$ | $g_2$ | $g_1$ | $g_2$ |
| | 1 | $L$ | - | - | 1 |
| $i$ | 2 | 1 | - | - | $L^2$ |
| | 3 | - | 1 | 1 | - |

Each trader is only interested in the good she does not possess, but only trader 2 has a large utility coefficient for any good. Thus, the social welfare will be dominated by the utility of the second trader.

For a single market, at the Walrasian equilibrium, we have $(L+1)p_1 = p_2$ (applying Claim 4 for trader 3), where $p_1$ and $p_2$ are the prices of $g_1$ and $g_2$, respectively. Therefore, traders 1 and 2 purchase $L/(L+1)$ and $1/(L+1)$ units of $g_2$, respectively, and trader 3 purchases $L+1$ units of $g_2$. The associated social welfare is

$$\frac{L}{L+1} + \frac{L^2}{L+1} + (L+1) = 2L+1$$

On the other hand if we cluster into 2 markets with trader 1 in one of the markets on her own, then trader 2 will purchase 1 unit of $g_2$ giving a social welfare of

$$0 + L^2 + 1$$

So, clearly, if we set $L$ to be arbitrarily large, the ratio between the social welfare with two markets and with a single market is unbounded.

**The Redistribution Ratio.**

Now consider the redistribution ratio. Clearly, for the linear exchange model, the maximum social welfare we can obtain via arbitrary welfare redistributions is $\sum_j (\max_i u_{ij}) \cdot (\sum_i e_{ij})$. This is generally not achievable by a Walrasian equilibrium even if we can cluster into an unlimited number of markets. In fact, redistribution ratio is also unbounded. Consider the following scenario with two traders and two goods $g_1, g_2$ and $g_3$. Let $L \gg 1$ be a large number. The endowments and utilities are given below.

| | | $\mathbf{e}_i$ | | $\mathbf{u}_i$ | |
|---|---|---|---|---|---|
| | | $g_1$ | $g_2$ | $g_1$ | $g_2$ |
| $i$ | 1 | 1 | - | - | $L^2$ |
| | 2 | - | 1 | 1 | $L$ |

The maximum social welfare is clearly more than $L^2$ as we could allocate all of good 2 to trader 1. At equilibrium in a single market, we have $p_1 = 1$ and $p_2 = L$ and thus trader 2 will keep $\frac{L-1}{L}$ units of good 2. This only leads to a social welfare of of

$$L^2 \cdot \frac{1}{L} + \left(L \cdot \frac{L-1}{L} + 1 \cdot 1\right) = 2L$$

Clearly, partitioning into more than one market cannot improve the situation here, so the optimal clustering is the single market. Thus, if we set $L$to be arbitrarily large then the ratio between the optimal social welfare and the maximum social welfare achievable by market clustering is unbounded.

## Appendix B: The Hardness of Market Clustering

### Clustering with a Bounded Number of Goods.

The market clustering problem in a linear exchange economy is NP-hard even when we have two markets and four goods.

**Theorem 8.** *Given an instance of the* 2*-market clustering problem with five goods and linear utility functions, it is NP-hard to decide whether there is a clustering that yields a social welfare of value at least $Z$, for any $Z > 0$.*

*Proof.* We give a reduction from the PARTITION problem, where we are given a set of $n$ integers $A = \{a_1, a_2, \dots, a_n\}$, and the goal is to decide whether there is a subset $A' \subseteq A$ such that $\sum_{a_i \in A'} a_i = \sum_{a_i \in A - A'} a_i = \frac{1}{2} \sum_{a_i \in A} a_i$. We may assume $a_i > 0$ for all $i$, as PARTITION is NP-complete in this case [20]. Also, we assume wlog that $S \geq 5$ since we can multiply every integer in $A$ by a positive integer.

Our key idea in the proof is to design an economy that can be clustered into two markets with a total social welfare of $f(x) = x/(x+1) + (c-x)/(c-x+1)$. The function $f$ attains its maximum over $[0, c]$ at $x_0 = c/2$. The reduction will follow, provided we can ensure that the market clustering of traders corresponding to $x$ also corresponds the optimal partition of the integers.

Towards this goal, consider the following economy with $n+4$ traders and 5 goods. Again, the LHS of the table show the initial endowments and the RHS shows the utility coefficients. We take $K \gg \sum_i a_i \gg 1$.

| | | $\mathbf{e}_i$ | | | | | $\mathbf{u}_i$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ |
| | $i \in \{1, \dots n\}$ | $a_i$ | $a_i$ | - | - | - | - | - | - | - | $1+\delta$ |
| | $n+1$ | 1 | 1 | $K$ | - | - | - | - | - | 1 | 1 |
| $i$ | $n+2$ | - | - | - | 1 | 1 | 1 | - | $K$ | - | - |
| | $n+3$ | 1 | 1 | - | $K$ | - | - | - | 1 | - | 1 |
| | $n+4$ | - | - | 1 | - | 1 | - | 1 | $-$ | $K$ | - |

**Claim 14.** *To optimise welfare, traders $n+1$ and $n+2$ must be in one market and traders $n+3$ and $n+4$ in the other market.*

*Proof.* Suppose traders $n+1$ and $n+2$ are in the first market and traders $n+3$ and $n+4$ are in the second market. Consider good $g_3$. Only traders $n+2$ and $n+3$ want $g_3$. So trader $n+2$ will obtain all $K$ units of trader $n+1$'s endowment of $g_3$ because trader $n+3$ is not present in their market. This gives trader $n+2$ a utility of at least $K^2$. Similarly with respect to good $g_4$, trader $n+4$ will receive a utility of at least $K^2$ in the other market. Thus, social welfare will be at least $2K^2$ with such a partitioning. The existence of such equilibria (regardless of the cluster assignment of traders 1 to $n$) follows from Gale's characterisation.

Now consider the other ways to cluster. There are $2K+8+2\sum_i a_i$ units of goods in total, so the maximum welfare of traders 1 to $n+1$ and $n+3$ is at most $(1+\delta)(2K+8+2\sum_i a_i) \ll 3K$. Consequently, to achieve a high welfare, traders $n+2$ and $n+4$ must do well.

Thus, the other possibility is if traders $n+1$ to $n+4$ are all in the same market. Let $\mathcal{A} := \sum_{i \in S} a_i$ where $S \subseteq \{1, \dots, n\}$ is the set of traders in the same market as traders $n+1$ to $n+4$. The market graph of the market is strongly connected, so the equilibrium prices are positive.

By symmetry, we have $p_1 = p_2$ and $p_3 = p_4$; we may fix $p_2 = 1$. Now only trader $n+4$ has a non-zero utility for the good $g_2$ so she must get it. Then by Claim 4, $\frac{1}{p_2} \geq \frac{K}{p_4}$, so $p_3 = p_4 \geq K$. We consider two cases:

(i) If $p_5 > p_3$, then traders $n+1$ and $n+3$ will not buy $g_5$. Thus only the traders 1 to $n$ will buy $g_5$. So, by Claim 1, $2p_5 = 2\mathcal{A}$. But then $p_5 \ll K \leq p_3$, a contradiction.

(ii) If $p_5 < p_3$, then traders $n+1$ and $n+3$ will spend all their money on good 5. Thus $2p_5 = (p_1 + p_2)(\mathcal{A} + 2) + (p_3 + p_4)K = 2\mathcal{A} + 4 + 2Kp_3 > 2p_3$, a contradiction.

This proves that $p_5 = p_3 = K$. From this we can compute the best bundle for any trader, whose utilities are given in this table:

| Trader | 1 to $n$ | $n+1$ | $n+2$ | $n+3$ | $n+4$ |
|---|---|---|---|---|---|
| budget | $2\mathcal{A}$ | $2+K^2$ | $2K$ | $2+K^2$ | $2K$ |
| welfare | $2\mathcal{A}(1+\delta)/K$ | $(2+K^2)/K$ | $2K$ | $(2+K^2)/K$ | $2K$ |

Hence this solution has low social welfare. □

So we may assume that traders $n+1$ and $n+2$ are in the first market and traders $n+3$ and $n+4$ are in the second market. The traders 1 to $n$ must be partitioned into these two markets. Let's say that a subset $S$ of the traders 1 to $n$ go in the first market and the other traders go into the second market.

We analyse the total welfare in Market 1. Let $\mathcal{A}_1 = \sum_{i \in S} a_i$ where $S$ is the set of indices of traders from $\{1, n\}$ in this market. As nobody wants $g_2$, we have $p_2 = 0$. The other goods are supplied and are desired so will have positive prices; fix $p_1 = 1$. At equilibrium, trader $n+2$ buys all of $g_1$ and $g_3$, so $p_3 = Kp_1 = K$, by Claim 4. So, applying Claim 1 for trader $n+2$ gives that $p_4 + p_5 = (\mathcal{A}_1 + 1)p_1 + Kp_3 = K^2 + \mathcal{A}_1 + 1$. From this the traders in $S$ receive $\mathcal{A}_1$ and trader $n+1$ receives $K^2 + 1$.

Trader $n+1$ must buy all of $g_4$ so, by Claim 4, we have $p_4 \leq p_5$. If $p_4 < p_5$, only traders 1 to $n$ would buy $g_5$ and therefore, by Claim 1, we obtain $p_5 = 1/\mathcal{A}_1$. So $p_4 + p_5 < 1/\mathcal{A}_1$, a contradiction. Hence $p_4 = p_5 = \frac{1}{2}(K^2 + \mathcal{A}_1 + 1)$. This gives the following welfares.

| Trader | 1 to $n$ | $n+1$ | $n+2$ |
|---|---|---|---|
| budget | $\mathcal{A}_1$ | $K^2+1$ | $K^2+\mathcal{A}_1+1$ |
| welfare | $2(1+\delta)\mathcal{A}_1/(K^2+\mathcal{A}_1+1)$ | $2(K^2+1)/(K^2+\mathcal{A}_1+1)$ | $K^2+\mathcal{A}_1+1$ |

Thus, the total welfare for the first market is

$$\frac{2(1+\delta)\mathcal{A}_1}{K^2+\mathcal{A}_1+1} + \frac{2(K^2+1)}{K^2+\mathcal{A}_1+1} + K^2 + \mathcal{A}_1 + 1 = 2\delta \cdot \frac{\mathcal{A}_1}{K^2+\mathcal{A}_1+1} + K^2 + \mathcal{A}_1 + 3$$

Symmetrically in Market 2, with $\mathcal{A}_2 := \sum_{i \notin S} a_i$, we get:

$$2\delta \cdot \frac{\mathcal{A}_2}{K^2+\mathcal{A}_2+1} + K^2 + \mathcal{A}_2 + 3$$

Over both markets the total welfare is

$$2\delta \cdot \left( \frac{\mathcal{A}_1}{K^2+\mathcal{A}_1+1} + \frac{\mathcal{A}_2}{K^2+\mathcal{A}_2+1} \right) + 2K^2 + \sum_i a_i + 6$$

Thus to optimise welfare over the two markets we need to choose $S$ to maximise

$$\frac{\sum_{i \in S} a_i}{1 + K^2 + \sum_{i \in S} a_i} + \frac{\sum_{i \notin S} a_i}{1 + K^2 + \sum_{i \notin S} a_i}$$

So we are trying to maximise a function a function $f$ defined by

$$f(x) = \frac{x}{1+K^2+x} + \frac{(c-x)}{1+K^2+(c-x)}$$

The function $f$ attains its maximum over $[0, c]$ at $x_0 = c/2$. Here $c = \sum_i a_i$ so this choice solves the partition problem. □

**Market Clustering with an Unbounded Number of Goods.**

In general, if the number of good is unbounded, the market clustering problem in a linear exchange economy is at least as hard as the maximum independent set problem. This holds even if we are allowed to partition the traders into as many markets as we wish (that is, $k = \infty$).

In the INDEPENDENT SET problem, we are given an undirected graph $G = (V, E)$, and the goal is to find a maximum cardinality set $S \subseteq V$ such that no two vertices of $S$ are adjacent in $G$. This problem is notoriously hard to approximate well.

**Theorem 15 ([14]).** *For any constant $\gamma > 0$, given a graph $G = (V, E)$ on $n$ vertices, unless* NP = ZPP*, it is hard to distinguish between the following two cases:*

- `Yes-Instance:` *The graph $G$ has an independent set of size $n^{1-\gamma}$.*
- `No-Instance:` *The graph $G$ has no independent set of size $n^{\gamma}$.* □

Here we present a reduction from INDEPENDENT SET that produces a similar hardness result for the market clustering problem (the case $k = \infty$):
**Theorem 9.** *For any constant $\delta > 0$, unless* NP = ZPP*, it is hard, considering an instance of the market clustering problem with linear utility functions, to distinguish between the following two cases:*

- `Yes-Instance:` *There is a clustering that yields a social welfare of value at least $Z^{1-\delta}$.*
- `No-Instance:` *There is no clustering that yields a social welfare of value at least $Z^{\delta}$.*

*where $Z$ is the maximum social welfare.*

*Proof.* Take an instance $G = (V, E)$ of INDEPENDENT SET with $V = \{v_1, \ldots, v_N\}$. Set the parameter $\gamma$ in Theorem 15 to $\gamma = \delta/2$. We now build an instance of the $k$-market clustering problem with $N + 1$ traders and $N + 1$ goods. For each $i \in \{0, \ldots, N\}$, the endowment of the trader $i$ is $e_{ii} = 1$, $e_{ij} = 0$ for any $j \neq i$. Thus, every trader has a unique specific good. The utility functions $u_i$ are defined with respect to the edge set. Specifically, for any trader $i$, with $i = 1, 2, \ldots, N$, set

$$u_{i0} = \varepsilon^2$$
$$u_{ij} = \begin{cases} \varepsilon \text{ if } \{v_i, v_j\} \in E \\ 0 \text{ otherwise} \end{cases}$$

Here, we choose $\varepsilon \ll 1/N$. The utility $u_0$ of trader 0 is defined as follows.

$$u_{0j} = 1 \quad \text{for all } j = 1, 2, \ldots, N$$
$$u_{00} = 0$$

`Yes-Instance:` Suppose the graph $G$ has an independent set $S$ of size $N^{1-\gamma}$. Then there is a clustering yielding a social welfare of $N^{1-2\gamma}$. To see this, place trader 0 and all the traders corresponding to nodes of $S$ in the first market; put the remaining traders in the second market. Consider the first market. This has an equilibrium by Gale's characterisation. Since $S$ is an independent set, trader 0 is the only trader who desires the goods held by traders in $S$. Thus, at equilibrium, the traders in $S$ are willing to trade with trader 0 but not amongst themselves. Therefore, trader 0 has utility of $N^{1-\gamma}$. Hence, the social welfare of the first market is at least $N^{1-\gamma} > N^{1-2\gamma}$.
`No-Instance:` Suppose the graph $G$ has no independent set $S$ of size $N^{\gamma}$. We will show that there is no clustering that yields a social welfare of $N^{2\delta}$. First, observe that the total social welfare incurred by the traders $1, \ldots, N$ is at most $\varepsilon N + \varepsilon^2 \ll 1$. We set $p_0 = 1$.

Next, consider the social welfare of trader 0. Let the equilibrium allocation and prices be $\mathbf{x}$ and $\mathbf{p}$, and consider the *trading graph* $H = (\{0, \ldots, n\}, A)$, with $A = \{ij \ : \ i, j \in V(H), x_{ij} > 0\}$. Take the component $K$ of the trading graph containing trader 0. By Claim 1 and Claim 2, the mapping $f : E' \to \mathbb{R}_+$ defined by $f(ij) = p_j x_{ij}$ is a non-zero circulation. Thus, the component $K$ must be strongly connected. Hence there is a closed walk $i_0 i_1, \ldots, i_{l-1} i_l$ with $i_0 = i_l$ and $x_{i_{j-1} i_j} > 0$ for all $j \in \{1, \ldots, l\}$ visiting every arc of $A$ at least once.

Now create an auxiliary set of prices $p'$ where $p'_i = p_i$ for all $i \neq 0$, and set $p'_0 = p_0/\varepsilon$. Observe that any trader buys goods only from amongst its neighbours with the minimum auxiliary price. We may assume that $p'_{i_0} \geq p'_{i_j}$ for any $j \in \{1, \ldots, l\}$. For any $j$, trader $j+1$ (indices taken modulo $l$) had the option to buy from trader $j$ and from trader $j+2$, and chose to buy from the latter. Thus, $p'_{i_j} \geq p'_{i_{j+2}}$ by the optimality constraint. This implies that the auxiliary prices can take at most two different values, and exactly one if $l$ is odd.

If the auxiliary prices take on only one value, then the price of any vertex trading with trader 0 must be $1/\varepsilon$, and the welfare of trader 0 is only $\varepsilon \ll 1$.

If the auxiliary prices take on two values then $K$ is bipartite. Let the bipartition be $A, B$ with $v_0 \in B$. Then $p'_b = 1/\varepsilon$ for any $b \in B$. If $A$ induces a stable set in $G$ then trader 0 can purchase at most $|A| \leq N^\gamma$ and so has utility at most $N^\gamma$. On the other hand, if $A$ induces at least one edge $ij$ in $G$ then $i$ had the option of buying from $j$ rather than from vertices in $B$. Thus $p'_a = p'_j \geq p'_b = 1/\varepsilon$. But $p_0 = 1$ so trader 0 can then afford at most $\varepsilon$ units of good from traders in $A$. Thus his utility is at most $\varepsilon$.

Consequently, the maximum utility of the traders is at most $1 + N^\gamma \leq N^{2\gamma}$ for large $N$. □

## Appendix C: Approximate Walrasian Equilibria

**Theorem 10.** *For $\varepsilon > 0$, every market has a market-clearing $\varepsilon$-approximate equilibrium.*

*Proof.* The proof is by induction on the number of goods $g$. By Gale's condition, a market with no altruist-free super self-sufficient sets admits an (exact) equilibrium, in particular markets with only one good admit equilibria.

Consider a market $M = (m, n, \mathbf{u}, \mathbf{e})$, and assume the existence of an altruist-free super self-sufficient set. Choose a minimal altruist-free super self-sufficient set $S \subset \{1, \ldots, n\}$. Let $G_S := \{j \in \{1, \ldots, m\} \ : \ \exists i \in S, e_{ij} > 0\}$ be the set of goods owned by traders in $S$. Let $H \subset G_S$ be the set of goods $j$ in $M_S$ such that $\sum_{i \in S} e_{ij} > 0$ but $\sum_{i \in S} u_{ij} = 0$. These are the goods relevant to the second condition of the definition of super self-sufficient sets, hence $H \neq \emptyset$. Let $M_S$ be the restriction of market $M$ to goods $G_S - H$ and traders $S$. Then the market $M_S$ has no super self-sufficient set by minimality of $S$. Hence there is an exact equilibrium $\mathbf{p}^S, \mathbf{x}^S$ on $M_S$.

Notice that $H$ is a proper subset of $G_S$, as $S$ is altruist-free. Define the market $M_T$ with traders $T := (\{1, \ldots n\} - S) \cup \{0\}$, and goods $G_T := (\{1, \ldots, m\} - G_S) \cup H$. The new trader 0 is altruistic ($\mathbf{u}_0 = 0$) with endowments $e_{0h} = \sum_{i \in S} e_{ih}$ for any $h \in H$, and $e_{0j} = 0$ for any $j \in G_T - H$. By induction on the number of goods, there is a market-clearing $\varepsilon$-approximate equilibrium $\mathbf{p}^T, \mathbf{x}^T$ on market $M_T$.

Let $A$ be the set of altruistic traders in market $M_T$. In particular, $0 \in A$. Define $P := \sum_{i \in A} \mathbf{p} \cdot \mathbf{e}_i$ to be the total budget of altruistic traders in $M_T$. Let $Q^S = \max_{i \in S} \mathbf{p}^S \cdot \mathbf{e}_i$

be the maximum budget of a trader in $M_S$. By scaling the prices $\mathbf{p}^S$, we may assume that:

$$\varepsilon Q^S \geq P \tag{5}$$

$$\frac{p_k^S}{u_{ik}} > \frac{p_j}{u_{ij}} \quad \text{for any trader } i \in T \text{, goods } j \in G_T \text{ and } k \in G_S \text{ with } x_{ij} > 0. \tag{6}$$

Our goal is to merge the two markets $M_S$ and $M_T$, and Inequality (6) will ensure that any non-altruistic trader in $M_T$ will not have a better response after merging. Inequality (5) will be used to prove the $\varepsilon$-optimality of the responses of traders in $S$.

We define a price allocation $\mathbf{p}$ and goods allocation $\mathbf{x}$ at equilibrium for the market $M$.

$$p_j = \begin{cases} p_j^T \text{ if } j \in G_T \\ p_j^S \text{ if } j \in G_S - H \end{cases}$$

For a non-altruistic trader $i$ in $T$, its allocation in $M$ is $\mathbf{x}_{ij} = \mathbf{x}_{ij}^T$ for any good $j$ in market $T$, $x_{ij} = 0$ otherwise. This allocation is a best response for trader $i$ at prices $p$ by Inequality (6) .

For an altruistic trader $i \neq 0$ in $M_T$, set $x_i$ to be any best response for $i$ in $M, \mathbf{p}$. In particular, $x_{ij} > 0$ implies that $j \in G_S - H$. For any good $j$ in $G_S - H$, let $\zeta_j := \sum_{i \in A} x_{ij}$ be the total quantity of good $j$ claimed in $M$ by altruistic traders of $M_T$, and for any good $j$ in $H$, let $\rho_j := \sum_{i \in A} x_{ij}^T$ be the total quantity of good attributed in $M_T$ to altruists. Notice that for any $j \in G_S - H$, $\zeta_j \leq \varepsilon \sum_{i=1}^n e_{ij}$ by Inequality (5).

For a trader $i \in S$ and a good $j \in G_S$, set

$$x_{ij} = \left(1 - \frac{\zeta_j}{\sum_{k=1}^n e_{kj}}\right) x_{ij}^S \geq (1 - \varepsilon) x_{ij}^S$$

Because $x_{ij}^S$ was a best exact response, $x_{ij}$ (restricted to $j \in G_S$) is an approximate best response in $M, \mathbf{p}$. Also, we have for any $j \in G_S$:

$$\begin{aligned} \sum_{i=1}^n x_{ij} &= \sum_{i \in S} \left(1 - \frac{\zeta_j}{\sum_{k=1}^n e_{kj}}\right) x_{ij}^S + \sum_{i \in A} x_{ij} \\ &= \sum_{i \in S} x_{ij}^S + \sum_{l \in A} \left(1 - \frac{\sum_{i \in S} x_{ij}^S}{\sum_{k=1}^n e_{kj}}\right) x_{lj} = \sum_{i \in S} x_{ij}^S \end{aligned}$$

Hence, the market is clearing on $S$.

For a trader $i \in S$ and a good $j \in G_T - H$, we set

$$x_{ij} = \frac{p^S \cdot x_i^S - \sum_{k \in G_S - H} p_k x_{ik}}{p^S \cdot \zeta} \rho_j$$

This is a way to redistribute the goods previously claimed by altruistic traders to traders in $S$ who "gave" part of their claims to altruists, in order to reach exact market-clearing. Then, for $j \in G_T - H$ we have:

$$\sum_{i \in S} x_{ij} = \frac{p^S \cdot x^S - \sum_{i \in S} \sum_{k \in G_S - H} p_k x_{ik}}{p^S \cdot \zeta} \rho_j = \frac{p^S \cdot (x^S - \sum_{i \in S} x_i)}{p^S \cdot \zeta} \rho_j = \rho_j$$

Moreover, the utility of good $j \in G_T - H$ for trader $i \in S$ is zero, hence $x_i$ is an approximate best response. □

## Appendix D: Limits of equilibria are generalized equilibria

**Theorem 11.** *For any market, any limit allocation $\mathring{\mathbf{x}}$ gives a generalized equilibrium.*

*Proof.* Let $(\mathbf{x}^n)_n$ be a sequence of market-clearing approximate equilibria in a market $M$, with approximation factor converging to 0, such that $(\mathbf{x}^n)_n$ converges to $\mathring{\mathbf{x}}$. For any $n$, let $\mathbf{p}^n$ be the set of prices supporting $\mathbf{x}^n$ in a market-clearing approximate equilibrium. We choose $(x^n)_n$ such that $p^n_j$ converges monotonically in $\mathbb{R}_+ \cup \{+\infty\}$.

Again, we may assume that for any good $g_j$, there is a trader $i$ that desires the good ($u_{ij} > 0$). Otherwise $p^n_j = 0$ for any $n$ and this good could have a generalized price $(0, 0)$. Consequently, for any good $g_j$, we have $p^n_j > 0$ for all $n \geq 0$ such that $\mathbf{x}^n$ is an $\varepsilon$-approximate equilibrium for $\varepsilon < 1$.

Let $\mathcal{C}$ be the partition of the vertices of the market graph into strongly connected components.

**Claim 16.** *Let $C \in \mathcal{C}$ be some component. Scale the prices $(\mathbf{p}^n)_n$ such that the maximum price of any good in $C$ is always 1. Then the sequence of prices of goods in $C$ converges to a set of positive prices.*

*Proof.* Otherwise, there is at least one good in $C$ whose price converges to zero. Since $p^n_{\max} = 1$, by strong connectivity, there is an arc $(j, j')$ with $p^n_j \to \gamma > 0$ and $p^n_{j'} \to 0$ when $n$ tends to $+\infty$. Let $i$ be a trader with $e_{ij} \geq e_{\min} > 0$ and $u_{ij'} \geq u_{\min} > 0$. Then as $n$ tends to $+\infty$, the ratio $u_{ij}/p^n_{j'}$ tends to $+\infty$, while the budget of $i$ tends to some constant larger than $\mathbf{e}_{\min} \cdot \gamma$, hence the welfare of $i$ tends to $+\infty$, contradiction. □

Denote $C \sim C'$ for $C, C' \in \mathcal{C}$ if there are $j \in C$, $j' \in C'$ such that $p^n_j/p^n_{j'}$ converges to a non-zero value. Then it is true for any $j \in C$, $j' \in C'$ by Claim 16, and $\sim$ is an equivalence relation. Let $\mathcal{J}_1, \ldots \mathcal{J}_l$ be the equivalence classes, and denote $J_i = \bigcup_{C \in \mathcal{J}_i} C$. We can order $J_1, \ldots J_l$ such that if $p^n_j/p^n_{j'}$ converges to zero, $j \in J_h$ and $j' \in J_{h'}$, then $h < h'$. We denote $J^-_a = J_1 \cup \ldots J_{a-1}$ and $J^+_a = J_{a+1} \cup \ldots \cup J_l$.

For any $h \in \{1, \ldots, l\}$, scale the prices $(p^n)_n$ such that the minimum price of a good in $J_h$ is 1, then set the generalized price of any $j \in J_h$ to $\mathring{p}_j = (h, \lim_{n\to+\infty} p^n_j)$. We now prove that $\mathring{\mathbf{p}}, \mathring{\mathbf{x}}$ is a generalized equilibrium.

`Budget constraint`: Let $i$ be a trader and $j$ a good maximizing $\mathring{p}_j = (r_i, p_j)$. Scale the prices $(p^n)_n$ such that the minimum price in $J_{r_i}$ is 1. For any $\alpha > 0$, for $n$ sufficiently large, the budget of $i$ is upper-bounded by $\sum_{j'} e_{ij'} p^n_{j'} \leq (1+\alpha) \sum_{j' \in J_{r_i}} e_{ij'} p^n_{j'}$. As $p^n_{j'}/p^n_{j''}$ tends to 0 for any goods $j' \in J_{r_i}$ and $j'' \in J^+_{r_i}$, $x^n_{ij''}$ also tends to $\mathring{\mathbf{x}}_{ij''} = 0$. This proves that for any $a > r_i$: $\sum_{j:\pi_1(\mathring{p}_j)=a} \mathring{\mathbf{x}}_{ij} \pi_2(\mathring{p}_j) = 0$.

Moreover, $\sum_j e_{ij} p^n_j = \sum_{j \in J^-_{r_i}} e_{ij} p^n_j + \sum_{j \in J_{r_i}} e_{ij} p^n_j \geq \sum_{j \in J_{r_i}} x^n_{ij} p^n_j$ as $u_{ij} = 0$ when $j \in J^-_{r_i}$. Because $\sum_{j \in J^-_{r_i}} e_{ij} p^n_j$ tends to zero, $\lim_{n\to+\infty} \sum_{j \in J_{r_i}} e_{ij} p^n_j \geq \lim_{n\to+\infty} \sum_{j \in J_{r_i}} x^n_{ij} p^n_j$, that is $\sum_{j \in J_{r_i}} e_{ij} \pi_2(\mathring{p}_j) \geq \sum_{j \in J_{r_i}} \mathring{x}_{ij} \pi_2(\mathring{p}_j)$.

`Market clearing`: Let $j \in P_h$ be any good. Then $\sum_i x^n_{ij} = 1$, hence $\sum_i \mathring{x}_{ij} = 1 = \sum_i e_{ij}$.

`Optimality`: Let $i$ be any trader. We know that $\sum_{j \in J^-_{r_i}} u_{ij} = 0$. If $\sum_{j \in J_{r_i}} u_{ij} = 0$, then the maximum welfare achievable by $i$ is zero, so $\mathring{\mathbf{x}}_i$ is a best response. We may assume that there is a $j \in J_{r_i}$ such that $u_{ij} > 0$. For any $\varepsilon \in [0, 1/2]$, there is $n = n_\varepsilon$ large enough such that the ratio utility/price for any good in $J^-_{r_i}$ is at most $\varepsilon \cdot \min_{j \in J_{r_i}, u_{ij}>0} u_{ij}/p^n_j$, and the approximation factor of $\mathbf{x}^n, \mathbf{p}^n$ is at most $\varepsilon$. Hence

$$(1+2\varepsilon) \sum_{j \in J_{r_i}} u_{ij} x^n_{ij} \geq \mathbf{u}_i \mathbf{x}^n_i = \sum_{j \in J^-_{r_i}} u_{ij} x^n_{ij} + \sum_{j \in J_{r_i}} u_{ij} x^n_{ij} \geq (1-\varepsilon) \cdot \left( \max_{j \in J_{r_i}} \frac{u_{ij}}{p^n_j} \right) \cdot \sum_{j \in J_{r_i}} e_{ij} p^n_j$$

When $\varepsilon$ tends to 0 (and thus $n_\varepsilon \to +\infty$), we get $\sum_j u_{ij}\mathring{\mathbf{x}}_{ij} \geq \left(\max_{j\in J_{r_i}} \frac{u_{ij}}{\pi_2(\mathring{p}_j)}\right)\sum_{j\in J_{r_i}} e_{ij}\pi_2(\mathring{p}_j)$, and this is the optimality constraint for exact generalized equilibria. □