# The graphs with the max-Mader-flow-min-multiway-cut property

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**Abstract**

We are given a graph $G$, an independant set $\mathcal{S} \subset V(G)$ of *terminals*, and a function $w : V(G) \to \mathbb{N}$. We want to know if the maximum $w$-packing of vertex-disjoint paths with extremities in $\mathcal{S}$ is equal to the minimum weight of a vertex-cut separating $\mathcal{S}$. We call *Mader-Mengerian* the graphs with this property for each independant set $\mathcal{S}$ and each weight function $w$. We give a characterization of these graphs in term of forbidden minors, as well as a recognition algorithm and a simple algorithm to find maximum packing of paths and minimum multicuts in those graphs.

## 1 Introduction

Given a graph $G = (V, E)$, a set $\mathcal{S} \subset V$ with $|\mathcal{S}| \geq 2$ and inducing a stable set is called a set of *terminals*. An $\mathcal{S}$*-path* is a path having distinct ends in $\mathcal{S}$, but inner nodes in $V \setminus \mathcal{S}$. A set $\mathcal{P}$ of $\mathcal{S}$-paths, is a *packing of vertex-disjoint* $\mathcal{S}$*-paths* (since there is no risk of confusion, we will use the shorter term *packing of* $\mathcal{S}$*-paths* within this paper), if two paths in $\mathcal{P}$ do not have a vertex in common in $V \setminus \mathcal{S}$. We are looking for a maximum number $\nu(G, \mathcal{S})$ of $\mathcal{S}$-paths in a packing.

An $\mathcal{S}$*-cut* is a set of vertices in $V \setminus \mathcal{S}$ that disconnect all the pairs of vertices in $\mathcal{S}$ (that is a blocker of the $\mathcal{S}$-paths). We are looking for an $\mathcal{S}$-cut with a minimum number $\kappa(G, \mathcal{S})$ of vertices.

The following inequality holds for any graph $G$ and any $\mathcal{S} \subseteq V(G)$: $\nu(G, \mathcal{S}) \leq \kappa(G, \mathcal{S})$, as any $\mathcal{S}$-path intersects any $\mathcal{S}$-cut. Note that if $|\mathcal{S}| = 2$ the equality always holds, being Menger's vertex-disjoint undirected $(s, t)$-paths theorem. This paper deals with graphs for which $\nu(G, \mathcal{S}) = \kappa(G, \mathcal{S})$, for any set $\mathcal{S}$ of terminals. Actually, we try to characterize a stronger property associated with a weighted version of these two optimization problems. Consider the following system with variables $x \in \mathbb{R}_+^{V \setminus \mathcal{S}}$:

$$x(P) \geq 1 \text{ for every } \mathcal{S}\text{-path } P \tag{1}$$

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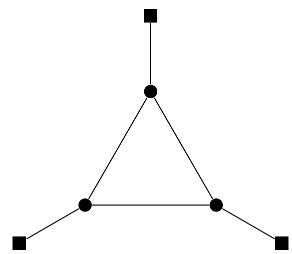

Figure 1: The net.

An integral vector $x$ minimizing $wx$ over (1) is necessarily a $0,1$-vector and is the characteristic vector of a minimum $\mathcal{S}$-cut. Dually, an integral vector $y$ optimum for the dual of minimizing $wx$ over (1) is necessarily a maximum $w$-packing of $\mathcal{S}$-paths. Hence, if (1) is a TDI system, we have that the minimum $w$-capacity of an $\mathcal{S}$-vertex-cut is equal to the maximum $w$-packing of $\mathcal{S}$-paths.

As an example, consider the graph of Figure 1, called *net*. Let $\mathcal{S}$ be the square vertices. A maximum integral packing of $\mathcal{S}$-paths ($w = 1$) contains only one path, while any $\mathcal{S}$-cut must contain at least two vertices. Precisely, there is a fractional packing of $\mathcal{S}$-paths of value $\frac{3}{2}$ (by taking each $\mathcal{S}$-path of length 3 with value $\frac{1}{2}$), and a fractional $\mathcal{S}$-cut with the same value (by taking $x(v) = \frac{1}{2}$ for all $v \notin \mathcal{S}$).

Motivated by the following property, we call *Mader-Mengerian* the graphs for which the system (1) is TDI for every set $\mathcal{S}$ of terminals.

*Property* 1. Given a graph $G$ and a set of terminal $\mathcal{S}$, the following conditions are equivalent:

1. The system (1) is TDI,
2. The polyhedron defined by (1) is integral,
3. The optimum value of maximizing $w^T x$ subject to (1) is integral (if finite) for all $w \in V^{\{0,1,+\infty\}}$.

The proof of this property is postponed to section 3 where the stronger Lemma 9 is proved. We already know that the net is not Mader-Mengerian.

Our main result (Theorem 10) is a description of the Mader-Mengerian graphs in terms of forbidden minors. However we do not use the usual minor operations (edge deletion and edge contraction), but *ad-hoc* operations on vertices. Our proof implies an algorithm (Lemma 9) to find maximal $w$-packing of paths in Mader-Mengerian graphs and minimum vertex multicuts for a given set of terminals. We also give a characterization of the pairs $(G, \mathcal{S})$ for which the system (1) is TDI (Theorem 15).

One of our most surprising results is that $G$ is Mader-Mengerian if and only if the system (1) is TDI for every independant set $\mathcal{S}$ of cardinality 3. This implies (with Lemma 9) a polynomial algorithm to recognize Mader-Mengerian graphs.

**Related results.**

Finding a minimum $\mathcal{S}$-cut is an NP-complete problem, even if $|\mathcal{S}| = 3$ [3]. In fact, [3] deals with edge-cuts (that is, sets of edges disconnecting $\mathcal{S}$), but one may observe that $\mathcal{S}$-edge-cut in a graph $G$ correspond to vertex-cut in the line-graph of the graph obtained from $G$ by adding one leaf to each vertex in $\mathcal{S}$.

Finding maximal packing of disjoint paths is a classical problem in graph theory, even if it was mainly studied for edge-disjoint (or arc-disjoint) paths. Menger [10] gave the first significant result, stating that when $|\mathcal{S}| = 2$, the maximum number of disjoint $S$-paths is equal to the minimum cardinality of an $(s,t)$-cut, both in edge-disjoint and vertex-disjoint cases. This result was further developped by Ford and Fulkerson [4], into what became the network flow theory. When there is more than two terminals, the results are however closer to matching theory than to network flows. Gallai [5] first proved a min-max theorem for packing of fully-disjoint $\mathcal{S}$-paths (that is even the ends of the paths must be disjoint), and his result was then strengthened by Mader [9] for fully-disjoint paths with ends in different parts of a partition of the terminals. Each part of the partition can be considered as one vertex of our set $\mathcal{S}$ in our setting, hence this implies a min-max theorem for the maximum number of $\mathcal{S}$-paths. We switch to this more convenient notation to state Mader's theorem. Considering a partition $\mathcal{T} = T_1, \dots, T_k$ of the terminals, let $\nu(G, \mathcal{T})$ be the maximum number of fully-disjoint $\mathcal{T}$-paths, where in this context, $\mathcal{T}$-path means a path whose ends are in different parts of $\mathcal{T}$. (In the following, we use $\mathcal{T}$ when considering fully-disjoint paths between a partition and $\mathcal{S}$ for inner disjoint paths between an independant set, both notions being reducible to one another.)

**Theorem 1** (Mader, 1978)**.** *Let $G$ be a graph and $\mathcal{T}$ a partition of the terminals of $G$. Then,*

$$\nu(G, \mathcal{T}) = \min |U_0| + \sum_{i=1}^{k} \left\lfloor \frac{b_{U_0}(U_i)}{2} \right\rfloor$$

*where the minimum ranges over all the partitions $U_0, \dots, U_k$ such that each $\mathcal{T}$-path intersects either $U_0$ or $E(U_i)$ for some $1 \le i \le k$. Here, $b_{U_0}(X) := |\{v \in X \ : \ v \in \mathcal{T} \text{ or } N(v) \setminus (X \cup U_o) \neq \emptyset\}|$.*

In the light of Mader's theorem, we are looking for graphs that admit a much simpler characterization: $\nu(G, \mathcal{S}) = \min |U|$ where the minimum ranges over sets $U$ such that each $S$-path intersects $U$.

Mader's theorem has been recently extended by Chudnovsky et al. [2], and by Gyula Pap [11].

Let us mention a similar result for edge-disjoint paths, that was proved by Cherkasky [1] and Lovász [8]:

**Theorem 2** (Cherkasky, Lovász, 1977)**.** *For any inner Eulerian graph $G$, then the maximum number of edge-disjoint $\mathcal{S}$-paths is equal to $\frac{1}{2}\sum_{s\in\mathcal{S}}\lambda s$, where $\lambda s$ is the minimum cardinality of a cut between $s$ and $\mathcal{S}-s$.*

This has been later extended by Karzanov and Lomonosov [7], who proved the Locking Theorem. These results explains when the maximum packing of edge-disjoint $\mathcal{S}$-paths has a characterization in terms of minimal cuts.

**Half-integrality of the vertex multiway cut problem.**

The problem of finding a minimum $\mathcal{S}$-cut is known to be 2-approximable. Indeed, Garg et al. [6] proved the stronger property that the relaxation based on equations (1) is half-integral. An even stronger property can be derived from an observation of Sebő and Szegő [12]. They reformulate Mader's theorem in the following way.

**Theorem 3** (Mader's theorem)**.**

$$\nu(G,\mathcal{T}) = \min_{\mathcal{X}} |X_0| + \sum_{C\in\mathcal{C}} \left\lfloor \frac{1}{2}|C\cap X| \right\rfloor$$

*where $\mathcal{X}$ ranges over all the families $X_0,\dots X_k$ of disjoint subsets of $V$ such that $T_i \in X_0 \cup X_i$ for all $1 \le i \le k$, and with $X = \bigcup_{i=1}^{k} X_i$ and $\mathcal{C}$ are the vertex-sets of the connected components of $G \setminus X_0 \setminus \bigcup_{i=1}^{k} E[X_i]$.*

The following is a remark from Andràs Sebő. This formulation implies that (1) is half-TDI. Indeed, a cost function for the minimum $\mathcal{T}$ cut problem implies a capacity function on the vertices for the dual problem, which is finding a packing of $\mathcal{T}$-paths. This capacity function can be easily implemented by replicating the vertices. To prove the half-TDIness, we need to show that when the capacity function is even, there is an optimal packing of $\mathcal{T}$-paths that is integral. However, it is easy to show that the vertices obtained by replicating a given vertex will all be in the same set $X_i$ for some $i$, or out of these sets. Hence the term $|C\cap X|$ will always be even and the *floor* operation can be removed from right-hand side. But then, the new expression gives also a bound for the *fractional* solution, and thus the fractional and integral optimum have the same value.

## 2 Vertex minors and skew minors

Given a graph $G=(V,E)$ and $v\in V$, *deleting* $v$ in $G$ means considering the graph $G-v$ induced by $V-v$, that is:

$$G - v := (V - v, E \setminus \delta_G(v))$$

*Contracting* $v$ means considering the graph $G/v$ obtained by removing $v$ and replacing its neighborhood by a clique:

$$G/v := (V - v, E \cup \{wx | w, x \in N_G(v)\} \setminus \delta_G(v))$$

For $e = xy \in E$ *contracting* $e$ means considering the graph $G/e$ obtained by identifying the end-nodes $x$ and $y$ of $e$.

$$G/e := (V, E \cup \{xz | z \in N_G(y)\} \cup \{yz | z \in N_G(x)\} \setminus e)$$

A graph obtained from $G$ by any sequence of vertex deletions and vertex contractions is a *vertex-minor* of $G$. A graph obtained from $G$ by any sequence of vertex deletions, vertex contractions and edge contractions is a *skew-minor* of $G$.

Vertex-minors can also be described in the following way:

**Proposition 4.** *Let $G$ be a graph, and $G'$ be a vertex-minor of $G$. Let $D$ be the vertices deleted and $C$ be the vertices contracted to get $G'$ from $G$. Then, $u, v \in V(G')$ are adjacent in $G'$ if and only if there is a path with extremities $u$ and $v$ in $G$ and whose inner nodes are in $C$.* □

This immediately implies:

**Lemma 5.** *Vertex-deletions and vertex-contractions commute.* □

By definition, for a class of graph, being closed under skew minors implies being closed under vertex minors, which in turn implies being closed under induced subgraphs. Several important classes of graphs are closed under skew minors. Among them:

**Definition 6.**
- *The* interval graphs *are the graphs of intersection of intervals of the real line.*
- *The* chordal graphs *are the graphs of intersection of subtree of a tree. Equivalently, a graph is chordal if each of its cycles of length at least* 4 *has a chord.*
- *The* cocomparability graphs *are the graphs whose complement is the underlying graph of a partially ordered set.*
- *The* Asteroidal-Triple-free (AT-free) graphs *are the graphs without asteroidal triple. A stable set $S$ of cardinality* 3 *is an* asteroidal triple *of $G$ if there is no $x \in S$ such that $S - x$ is contained in a connected component of $G - (x \cup N(x))$.*
- *The $P_k$-free graphs, for $k \in \mathbb{N}$, are the graphs with no induced path of length at least $k$.*

The following proposition is left as an exercise:

**Proposition 7.** *Interval graphs, chordal graphs, co-comparability graphs, AT-free graphs, $P_k$-free graphs are closed under skew minors.* □

The following lemma explains why we are interested in the vertex-minor operations.

**Lemma 8.** *Given a graph $G$ and a set of terminal $\mathcal{S}$, if the system* (1) *is TDI, then it is also TDI for any vertex-minor of $G$.*

*Proof.* Deleting $v \in V \setminus \mathcal{S}$ corresponds to setting $w_v = 0$. Contracting $v \in V \setminus \mathcal{S}$ corresponds to setting $w_v = +\infty$. □

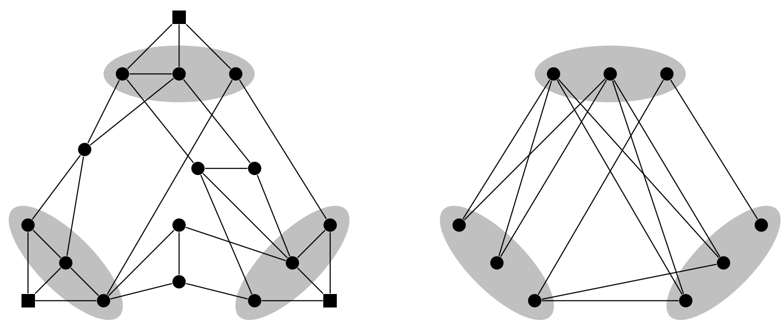

Figure 2: A graph $G$ and the auxiliary graph $G_{\mathcal{S}}$.

## 3 Integrality of the blocker of S-paths

For a given graph $G$ and a set $\mathcal{S}$ of terminal, we construct an auxiliary graph $G_{\mathcal{S}}$ as follows. First, note that if a non-terminal vertex $v$ is adjacent to two terminals $s$ and $t$, we may assume that the maximum packing for a weight function $w$ contains $w(v)$ times the 2-length paths $sv, vt$, and the minimal $\mathcal{S}$-cut contains $v$. Hence, we first delete every non-terminal vertex adjacent to two or more terminals.

We may also assume that no $\mathcal{S}$-path of a maximum packing contains two vertices of $N_G(s)$ for some terminal $s$ (by taking chordless paths). Therefore if $G - N_G(s)$ contains a component disjoint from $\mathcal{S}$, we can delete all its vertices.

From now, we will always suppose that:

$(i)$ $G$ has no vertices adjacent to two distinct terminals.

$(ii)$ for each $s \in \mathcal{S}$, every component of $G - N_G(s)$ intersects $\mathcal{S}$.

$(iii)$ $G$ has no edge whose ends are both adjacent to the same terminal.

Then we consider the set $N = N_G(\mathcal{S})$ of vertices adjacent to $\mathcal{S}$. $N$ is the vertex set of $G_{\mathcal{S}}$. We delete the terminals, and contract the vertices in $V - (N \cup \mathcal{S})$. Then we remove all the edges whose ends are adjacent to the same terminal in $G$ (the contraction of a path of a maximum packing would not use these edges) . This gives $G_{\mathcal{S}}$. By construction, this graph is $|\mathcal{S}|$-partite, each part being the neighborhood of one terminal.

Note that $a, b \in N$ are adjacent in $G_{\mathcal{S}}$ if $a$ and $b$ are not adjacent to a common terminal, and there is an $(a, b)$-path in $G$ whose inner vertices are outside $\mathcal{S} \cup N_G(\mathcal{S})$.

**Lemma 9.** *Given a graph $G$ and a set of terminal $\mathcal{S}$, the system* (1) *is TDI if and only if the auxiliary graph $G_{\mathcal{S}}$ is bipartite.*

*Proof.* Assume that $G_{\mathcal{S}}$ is not bipartite. Let $C^*$ be an induced odd cycle of $G_{\mathcal{S}}$. We define a weight vector $w \in V(G)^{\{0,1,+\infty\}}$ as follows:

$$w_v := \begin{cases} 1 & \text{if } v \in C^* \\ 0 & \text{if } v \in V(G_{\mathcal{S}}) \setminus C^* \\ +\infty & \text{otherwise} \end{cases} \tag{2}$$

To every edge $uv$ of $C^*$, we can associate an $\mathcal{S}$-path of $G$ intersecting $N$ exactly in $u$ and $v$. Then a maximum fractional $w$-packing of $\mathcal{S}$-paths is given by taking $1/2$ for each of these paths and a minimum fractional $\mathcal{S}$-cut of $G$ is given by $1/2$ on every node of $C^*$, and 1 on other vertices of $N$. The optimum value of the corresponding pair of dual linear programs is then $|V(C^*)|/2$, hence the polyhedron defined by (1) is not integer.

Suppose now that $G_{\mathcal{S}}$ is bipartite, with bipartition $(A, B)$.

Let $H$ be the graph obtained by deleting $\mathcal{S}$ and add two new non-adjacent vertices $s_a$ and $s_b$, adjacent to respectively $A$ and $B$.

Let $P$ be a chordless $(s_a, s_b)$-path in $H$. Let $\{a, b\} := N \cap V(P)$. We can associate a unique path $\hat{P}$ of $G$ to $P$, by replacing its extremities by terminals of $G$ (because each vertex of $N$ is adjacent to a unique terminal). We show that $\hat{P}$ cannot be a cycle. Let $Q = V(\hat{P}) \setminus (\mathcal{S} \cup N)$. If $Q$ is empty, $\hat{P}$ is clearly not a cycle because in $H$, the neighborhood of a terminal is a stable set.

Else $Q$ is contained in a component $C$ of $G \setminus (N \cup \mathcal{S})$. $C$ is adjacent to $N_G(s)$ and $N_G(t)$ for two distinct terminals $s$ and $t$ by condition $(ii)$. We can suppose that $a \in N_G(s)$. $\hat{P}$ is a cycle only if $b \in N_G(s)$. But if this was the case, then for $c \in N_G(t)$ adjacent to $C$, $a, c, b$ would be a path in $H$, hence $a$ and $b$ would be in the same part of the bipartition $(A, B)$, contradiction. $\hat{P}$ is not a cycle, it is an $\mathcal{S}$-path.

By applying the vertex-disjoint version of Menger's theorem to $H$, $\nu(G, w, \mathcal{S}) = \kappa(G, w, \mathcal{S})$ for any $w \in \mathbb{Z}^{V \setminus \mathcal{S}}$. □

## 4 A forbidden minor characterization

In this section, we find a characterization of Mader-Mengerian graphs by excluded vertex-minors. We start from the proof of Lemma 9, where we showed that if a graph is not Mader-Mengerian, its auxiliary graph has an odd cycle. In the auxiliary graph construction, we perform vertex-minor operations plus deletion of edges between two vertices adjacent to the same terminal. It follows that a graph that is not Mader-Mengerian contains a vertex-minor $G$ of the following form.

$G$ is a graph obtained by taking an odd cycle $C$ and the terminals adjacent to $C$. Each vertex of $C$ is adjacent to exactly one terminal, called the *representant* of this vertex. We color the vertices depending on their representants: each representant gets a distinct color, each other vertex has the color of its representant. A color is thus a set of vertices adjacent to some terminal, plus this terminal. Two consecutive vertices of the odd cycle have distinct colors, while the extremities of each chord share the same color. Let $\mathcal{A}_n$ be the class of graphs obtained in this way with $n$ terminals.

One path of lemmas and proofs to obtain the following result is presented in the Appendix.

**Theorem 10.** *Let $G$ be a graph.The system* (1) *is TDI for every stable set $\mathcal{S}$ if and only if $G$ does not contain a vertex minor in $\mathcal{A}_3$.*

*Proof.* Direct consequence of Lemmas 19 and 23. □

**Corollary 11.** *System* (1) *is TDI for every stable set $\mathcal{S}$ of $G$ if and only if it is TDI for every stable set $\mathcal{S}$ of cardinality* 3 *of $G$.* □

This gives a polynomial-time recognition algorithm for the related class of graphs, in combination with Lemma 9: we only have to check for each independant subset of three vertices whether the associated auxiliary graph is bipartite. Another important consequence is that the class of graphs for which system (1) is TDI for every stable set is large. Indeed, it contains at least the asteroidal-triple-free graphs:

**Corollary 12.** *For every asteroidal-triple-free graph, the system* (1) *is TDI.*

*Proof.* Follows from Theorem 10 and Proposition 7, as every graph in $\mathcal{A}_3$ contains an asteroidal triple, namely the set of terminals. □

To conclude this section on vertex-minors, we prove that there is an infinite number of minimal graphs to exclude.

**Lemma 13.** *If $n = 3$ and each color class induces a clique, then $G$ is a minimal excluded graph.*

*Proof.* Let $U, V, W \subset V(G)$ be the three colors of $G$, Let $u$, $v$, $w$ be the three terminals of a minimal excluded minor $G' = G - D/C$. The distance between two terminals in $G'$ is at least 3, in particular they cannot be adjacent. If $x, y \in V(G')$ and $xy \in E(G)$ then $xy \in E(G')$, thus $u$, $v$ and $w$ have distinct colors in $G$, say $u \in U$, $v \in V$, $w \in W$.

Let $U'$, $V'$ and $W'$ be the color classes of $u$, $v$ and $w$ respectively in $G'$. Every vertex adjacent to $u$ in $G'$ must be in the same color class $U'$ as $u$ in $G'$, proving that $U \setminus (D \cup C) \subset U'$. Because color classes are a partition of the vertex set, we have equality, $U' = U \setminus (C \cup D)$ and similary for $V'$ and $W'$.

Suppose $C$ is not empty, let $x \in C$. We may assume $x \in U$. If $x$ is the representant of $U$, then $G' = G - (D + x)/(C - x)$. Else, if $x$ is not the representant of $U$, $x$ has exactly two neighbors $y$ and $z$ outside $U$. Because $u$, $v$ and $w$ must be at distance 3 of each other in $G'$, $y, z$ must be in $D$. Then we also have that $G' = G - (D + x)/(C - x)$. Hence $G' = G - (C \cup D)$. But then, as the set of edges between colors of $G'$ must be a cycle, $G' = G$, proving that $G$ is minimal. □

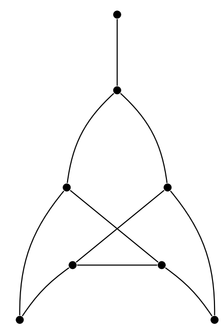

Figure 3: The rocket

## 5 Minimal skew-minors exclusion

A skew-minor of a graph $G$ is any graph obtained from $G$ via the following operations: vertex deletion, vertex contraction and edge contraction.

Note that Mader-Mengerian graphs are not closed under edge contraction since by inflating one of the central vertex of the net we get a Mader-Mengerian graph. However, we can get a simple sufficient condition for the integrality of system (1) based on skew-minors:

**Theorem 14.** *Any graph $G$ is either Mader-Mengerian or contains a net or a rocket as a skew minor.*

*Proof.* We show how to find a net or a rocket skew-minor from any non-Mader-Mengerian graph $G$, by induction on the number of vertices. First assume that $G$ is minimal under taking vertex minors so that it is in the class $\mathcal{A}_3$ (Theorem 10). Let $C$ be the odd cycle obtained from $G$ by deleting the three terminals. Second, assume that $G$ doesn't contain a monochromatic chord. Otherwise contract it, creating a new properly colored odd cycle using vertices and edges in $C$ only, and therefore a smaller non-Mader-Mengerian graph than $G$.

If all color classes contain at most two vertices, because $|V(C)|$ is odd, we have either $|V(C)| = 3$ and $G$ is the net, or $|V(C)| = 5$ and $G$ is the rocket.

Otherwise, one color class $\mathcal{C}$ of $G$ contains at least three vertices, let $P : u - v$ be a maximal odd path of $C$ induced by the two other color classes (Lemma 22, or there is a net). Let $u'$ and $v'$ be the two neighbours of $u$ and $v$ in $\mathcal{C}$. Let $w \in \mathcal{C} \setminus \{u', v'\}$ and $t$ be the terminal adjacent to $\mathcal{C}$. Contract edges $u't$ and $v't$. Delete all vertices except $w$, the three terminals and the vertices of $P$. The new contracted graph is still in $\mathcal{A}_3$. So we can apply induction. □

## 6 When the set of terminals is fixed

Our arguments apply when we want to find the pairs $(G, \mathcal{S})$, $\mathcal{S} \subset V(G)$, for which the system (1) is TDI. Up to now, we have only looked at graphs $G$ for which we have TDIness for every set of terminals. To deal with a fixed set of terminals, we define another notion of vertex-minor, the *signed vertex-minor*, defined on pairs $(G, \mathcal{S})$. Signed vertex-minor are defined like vertex-minor, except that the set of terminals of the minor must be a subset of the terminals of the original graph. More precisely, $(H, \mathcal{S}')$ is a signed vertex-minor of $(G, \mathcal{S})$ if $H$ is a vertex-minor of $G$ and $\mathcal{S}' \subseteq \mathcal{S}$.

Recall that $\mathcal{A}_3$ is the class of graphs built from a three-colored odd cycle, by adding a terminal for each color, and chords with extremities of the same color. We define similarly the class $\overline{\mathcal{A}_3}$ of signed vertex-minor $(G, \mathcal{S})$, where $G \in \mathcal{A}_3$, and $\mathcal{S}$ is the set of the three terminals in the construction of $G$.

This setting does not affect Lemma 9, and then the following theorem, close to Theorem 10, can be deduced by the same proof. Indeed, the proofs in Section 4 never create new terminals when considering vertex-minors, and hence are still valid for signed vertex-minors.

**Theorem 15.** *Let $G$ be a graph and $\mathcal{S}$ a set of terminal in $G$. The system* (1) *is TDI if and only if $(G, \mathcal{S})$ does not have a signed vertex-minor in $\overline{\mathcal{A}_3}$.* □

**Corollary 16.** *The system* (1) *is TDI for $(G, \mathcal{S})$ if and only if it is TDI for every $(G, \mathcal{S}')$, with $\mathcal{S}' \subseteq \mathcal{S}$, $|\mathcal{S}| = 3$.* □

Moreover, all the graphs of $\overline{\mathcal{A}_3}$ are minimal graphs by signed vertex-minors for which system (1) is not TDI. Indeed, a potential minor would have the same set of terminals. Moreover, if we contract a vertex, then its two consecutive vertices in the odd cycle become adjacent to two terminals, hence must be deleted. Hence, the minor must be obtained without vertex contraction, and the minimality follows easily.

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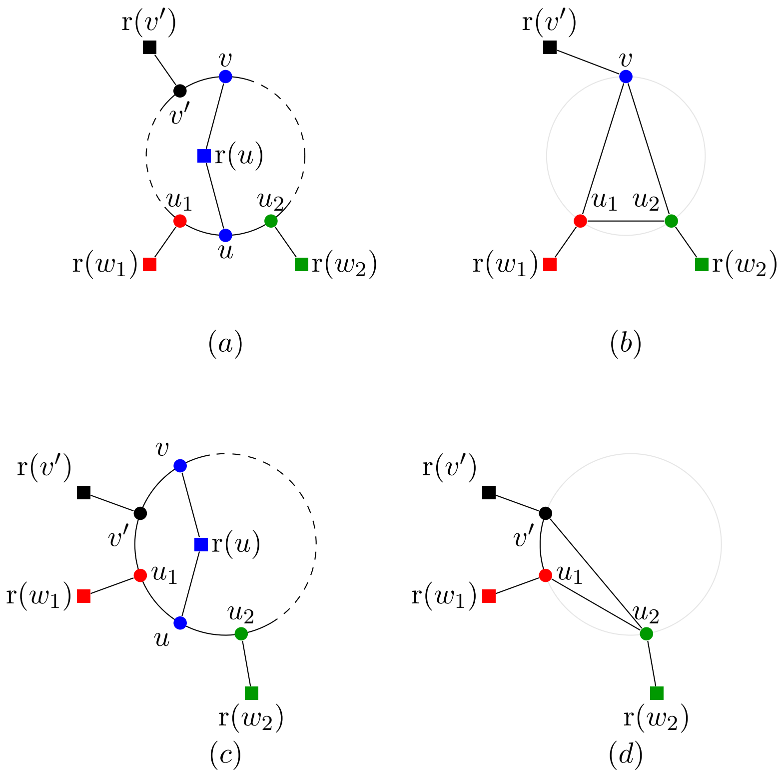


Figure 4: Illustration for Lemma 17.

## Appendix to section 4

The *distance in $C$* between two vertices $u$ and $v$ is the minimum number of arcs in one of the two $(u, v)$-paths in $C$. We denote $d_C(u, v)$ this minimum. We say that $u$ and $v$ are *consecutive* if $d_C(u, v) = 1$. We denote $\mathrm{r}(u)$ the representant of a vertex $u$. We say that a vertex of $C$ is *bicolored* if its two neighbors in $C$ have distinct colors. Two colors are *adjacent* if there is an edge in $C$ whose ends have these two colors.

Note that the net is a forbidden minor of Mader-Mengerian graphs, and is minimal. We try to find other forbidden minors that do not have a net as vertex-minor. For a graph $H$, we say that $G$ is $H$-free if $H$ is not a vertex minor of $G$.

**Lemma 17.** *Let $u$ be a bicolored vertex. Let $v$ be a vertex of the same color as $u$. Then, either $G$ contains a net, or every vertex consecutive to $v$ has the color of a vertex consecutive to $u$.*

*Proof.* Let $u_1$ and $u_2$ be adjacent to $u$ in $C$, $v'$ is adjacent to $v$, and $u_1$, $u_2$ and $v'$ have distinct

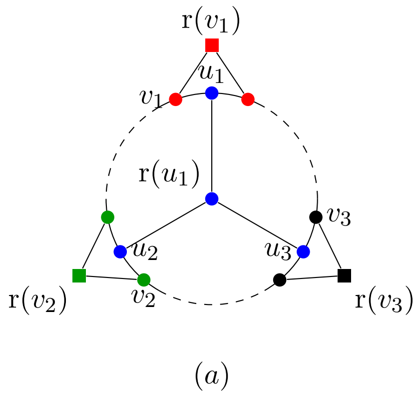


Figure 5: Illustration for Lemma 18.

colors. First, suppose that $d_C(u, v) \geq 3$. There are two cases.

If $d_C(v', u) \geq 3$ (Figure 4, $a$), let $G'$ be the graph obtained by contracting $u$ and $\mathrm{r}(u)$ and by deleting all the vertices except $\mathrm{r}(u_1)$, $\mathrm{r}(u_2)$, $u_1$, $u_2$ and $v$. $G'$ is a net (Figure 4, $b$).

If $d_C(v', u) = 2$ (Figure 4, $c$), we may assume $v'u_1 \in E(C)$. Let $G'$ be the graph obtained from $G$ by contracting $u$, $v$ and $\mathrm{r}(u)$ and deleting every other vertex except $v'$, $\mathrm{r}(v')$, $\mathrm{r}(u_1)$, $\mathrm{r}(u_2)$, $u_1$ and $u_2$. Then $G'$ is a net (Figure 4, $d$).

Now suppose that $d_C(u, v) = 2$. We may assume that $u_1$ is adjacent to $v$. Then the graph obtained from $G$ by contracting $\mathrm{r}(u)$ and deleting every vertex except $u$, $v$, $u_1$, $u_2$, $v'$ and $\mathrm{r}(u_1)$, is a net. □

**Lemma 18.** *Every color is adjacent to at most two other colors, or $G$ contains a net vertex-minor.*

*Proof.* Let $R$ be any color. By applying iteratively Lemma 17, if there is a vertex of color $R$ whose two consecutive vertices have distinct colors, then either $G$ contains a net, or $R$ is adjacent to exactly two colors.

Otherwise, each vertex in $C$ is consecutive to two vertices of the same color. Suppose that there are three vertices $u_1$, $u_2$, $u_3$ in $C$ of color $R$, such that their neighbors have three different colors. let $v_1$, $v_2$ and $v_3$ be the vertices following $u_1$, $u_2$, $u_3$ respectively in $C$ (Figure 5, $a$). Then, by contracting $u_1$, $u_2$, $u_3$, $\mathrm{r}(u_1)$ and deleting all the vertices except $v_1$, $v_2$, $v_3$ and their representants, we obtain a net (Figure 5, $b$). □

From now, we suppose that $G$ does not have a net minor. We define the *graph of colors*, whose vertices are the colors, by the adjacency relation introduced above. By Lemma 18, the graph of color has maximum degree two. By connexity, it is either a cycle or a path. We index the colors from 1 to $n$, following the order defined by the path or the cycle. Thus, each edge of $C$ has extremities of colors $i$ and $i + 1$, or 1 and $n$. We have the following immediate consequence.

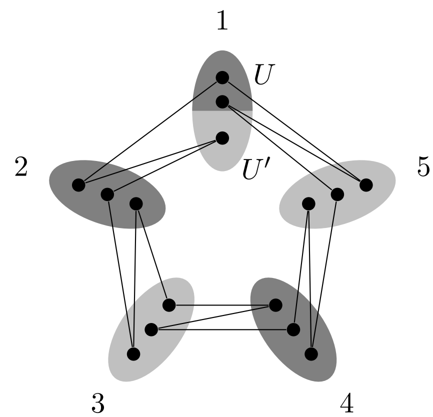


Figure 6: Illustration for Lemma 20, each ellipse represents a color.

**Lemma 19.** *Let $G$ be net-free. The number $n$ of colors is odd, and the graph of colors is a cycle.*

*Proof.* Suppose not. Then $C$ has a proper 2-coloring (following the parity of the colors), thus is even, contradicting the assumption. □

**Lemma 20.** *Let $G$ be net-free. Every color contains a bicolored vertex.*

*Proof.* Without loss of generality, it is sufficient to prove that there is a bicolored vertex of color 1. Let $U$ be the vertices of color 1 adjacent to a vertex of color $n$ and $U'$ be the other vertices of color 1. Let $W$ be the vertices of $C$ having an odd color minus $U$, and $B$ its complement in $V(C)$ (see Figure 6). By the definition of $U$, $W$ does not contain two consecutive vertices of $C$, and $(W, B)$ cannot be a proper two coloring of $C$. Thus there is an edge in $C$ with both extremities in $B$. But this can only be an edge between $U$ and color 2, hence there is a bicolored vertex in $U$. □

**Lemma 21.** *Suppose $G$ is net-free. The number of edges between any two color classes is odd.*

*Proof.* Choose two adjacent colors, and remove from $C$ every edge between these two colors. Every path thus obtained has its both extremities in the same color class, or in the two chosen colors. So every path has an even length. As $C$ is odd, this proves that we removed an odd number of edges. □

**Lemma 22.** *Suppose $G$ is net-free. There is a maximal sequence of consecutive edges in $C$ between any two given adjacent colors of length $2k + 1$, for some $k \geq 0$.*

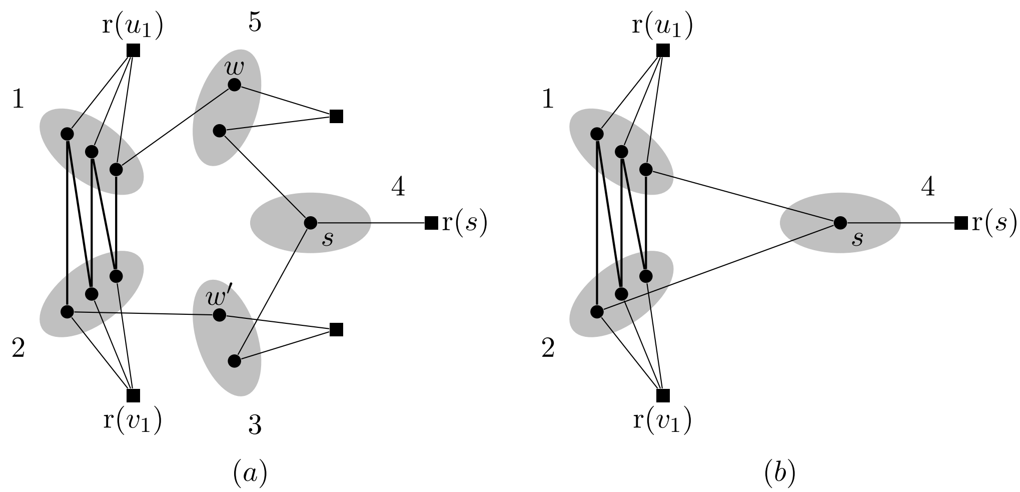


Figure 7: Illustration for Lemma 23

*Proof.* Consider the subpaths of $C$ obtained by keeping only the edges between colors 1 and 2. By Lemma 21, there is a path of odd length, proving the lemma. □

**Lemma 23.** *If $n \geq 5$, then $G$ is not a minimally excluded graph by vertex minor.*

*Proof.* If $G$ contains a net minor, it is clearly not minimal. Suppose it does not.

Let $u_1$, $v_1$, $u_2$, $\ldots u_k$, $v_k$ be a maximum subpath of $C$ of odd length between colors 1 (containing the $U_i$'s) and 2 (containing the $v_i$'s), with $k \geq 1$. Let $w$ be the vertex of color $n$ adjacent to $u_1$, and $w'$ the vertex of color 3 adjacent to $v_k$. Let $s$ be a bicolored vertex of color 4 (see Figure 7, *a*).

Consider the graph obtained by contracting vertices of colors 3 and $5, \ldots, n$, and deleting all the other vertices except $u_1$, $v_1$, $u_2$, $\ldots u_k$, $v_k$, $s$, $\mathrm{r}(u_1)$, $\mathrm{r}(u_2)$ and $\mathrm{r}(s)$. This graph (Figure 7, *b*) is composed of a cycle of length $2k+1$ plus three terminals $\mathrm{r}(u_1)$, $\mathrm{r}(u_2)$ and $\mathrm{r}(s)$. It obviously checks the condition for being an excluded graph. Thus $G$ is not minimal. □