Packing and covering with balls on Busemann surfaces

Victor Chepoi, Bertrand Estellon, and Guyslain Naves

Laboratoire d'Informatique Fondamentale, Aix-Marseille Université and CNRS,
Faculté des Sciences de Luminy, F-13288 Marseille Cedex 9, France
{victor.chepoi, bertrand.estellon, guyslain.naves}@lif.univ-mrs.fr

Abstract. In this note we prove that for any compact subset S of a Busemann surface $(\mathcal{S}, d)$ (in particular, for any simple polygon with geodesic metric) and any positive number δ, the minimum number of closed balls of radius δ with centers at $\mathcal{S}$ and covering the set S is at most 19 times the maximum number of disjoint closed balls of radius δ centered at points of S: $\nu(S) \leq \rho(S) \leq 19\nu(S)$, where $\rho(S)$ and $\nu(S)$ are the covering and the packing numbers of S by δ-balls.

1. Introduction

The set packing and the set covering problems are classical questions in computer science [34], combinatorics [5], and combinatorial optimization [19, 32]. Packing and covering problems in $\mathbb{R}^d$ with special geometric objects have been also actively investigated in computational geometry [1, 9, 14, 28] and in discrete geometry [7, 29]. Finally, the covering and packing problems of arbitrary metric spaces with balls (which is the subject of the current paper) have been formulated in the middle of 20th century in pure mathematics [25]. The respective covering and packing numbers capture the size of the underlying metric space and play a central role in several areas of pure and applied mathematics: information theory, functional analysis, probability theory, statistics, and learning theory [21, 26, 27].

In the *set covering problem*, given a collection $\mathcal{F}$ of subsets of a (finite or infinite) domain X, the task is to find a subcollection of $\mathcal{F}$ of minimum size $\rho(\mathcal{F})$ whose union is X. The *set packing problem* asks to find a maximum number $\nu(\mathcal{F})$ of pairwise disjoint subsets of $\mathcal{F}$. Another problem closely related to set covering is the hitting set problem. A subset T is called a *hitting set* of $\mathcal{F}$ if $T \cap S \neq \varnothing$ for any $S \in \mathcal{F}$. The *minimum hitting set problem* asks to find a hitting set of $\mathcal{S}$ of smallest cardinality $\tau(\mathcal{F})$. All these three problems are NP-hard, moreover, they are difficult to approximate within a constant factor unless $P = NP$. In case when X is a metric space and $\mathcal{F}$ is the set of its balls of equal radii, then one can easily see that the minimum covering and the minimum hitting set problems are equivalent, i.e., $\rho(\mathcal{F}) = \tau(\mathcal{F})$.

The inequality $\tau(\mathcal{F}) \geq \nu(\mathcal{F})$ holds for any family of sets $\mathcal{F}$ on any domain X: any two sets from a packing cannot be hit by the same point of X. Of particular importance are the families of sets $\mathcal{F}$ for which there exists a universal constant $c := c(\mathcal{F})$ such that $\tau(\mathcal{F}') \leq c\nu(\mathcal{F}')$ holds for any subfamily $\mathcal{F}'$ of $\mathcal{F}$. In general, proving that for all subfamilies of a particular family of sets $\mathcal{F}$ such a universal constant c exists is a notoriously difficult problem and it is open for many simple particular cases. For example, in 1965, Wegner [36] asked if for the family

$\mathcal{R}$ of all axis-parallel rectangles in $\mathbb{R}^2$ it is always true that $\tau(\mathcal{R}) \leq 2\nu(\mathcal{R}) - 1$ (Gyárfás and Lehel [23] relaxed this question by asking if $\tau(\mathcal{R}) \leq c\nu(\mathcal{R})$ for a universal constant c).

We briefly review now some families $\mathcal{F}$ for which the inequality $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ holds (when $\mathcal{F}$ is a family of balls in a metric space some known results will be reviewed in the next section). The equality $\tau(\mathcal{F}) = \nu(\mathcal{F})$ holds if $\mathcal{F}$ is an interval hypergraph, a hypertree, and more generally, a normal hypergraph [5, 32]. Covering and packing problems for special families of subtrees of a tree have been considered in [4, 32]. Alon [2, 3] established that if ${}^{\kappa}\mathcal{I}$ is a family of κ-intervals (unions of κ intervals) of the line (or a family consisting of unions of κ subtrees of a tree), then $\tau({}^{\kappa}\mathcal{I}) \leq 2\kappa^2\nu({}^{\kappa}\mathcal{I})$. A similar result has been obtained in [15] for unions of κ balls in a geodesic δ-hyperbolic space. Gyárfás and Lehel's relaxation of Wegner's conjecture was confirmed in [17, 20] for families of axis-parallel rectangles intersecting a common monotone curve. One common feature of all these results is that the inequality $\tau(\mathcal{F}) \leq c\nu(\mathcal{F})$ is established by constructing in a primal-dual way a hitting set T and a packing $\mathcal{P} \subseteq \mathcal{F}$ such that $|T| \leq c|\mathcal{P}|$. Consequently, this provides a factor c approximation algorithm for hitting set and packing problems for $\mathcal{F}$.

In this note, we consider the problem of covering and packing by balls of equal radii of subsets of Busemann surfaces. Using a similar approach as above, we prove that the minimum number of closed balls of radius δ required to cover a compact subset S of a Busemann surface $(\mathcal{S}, d)$ is at most 19 times the maximum number of pairwise disjoint closed balls of radius δ with centers in S. Our initial motivation was to establish that such an inequality holds for simple polygons with geodesic metric. Busemann surfaces represent a far-reaching generalization not only of simple polygons, but also of Euclidean and hyperbolic planes and of all planar polygonal complexes of global non-positive curvature. Roughly speaking, a Busemann surface is a geodesic metric space homeomorphic to $\mathbb{R}^2$ in which the distance function is convex [30].

2. Preliminaries and main results

In this section, we recall all necessary definitions and results related to the subject of this paper. We start with two subsections, one dedicated to basic notions and notations about covering and packing problems, and the second one to some known results on covering and packing metric spaces and graphs with balls. Then in two subsequent subsections we recall some definitions and notations on geodesic metric spaces and Busemann surfaces. We conclude the section with the formulation of the main results.

2.1. **Covering and packing with balls.** Let (X, d) be a metric space and let δ be an arbitrary positive real number. For a point $x \in X$, we will denote by $B_\delta(x) = \{y \in X : d(x, y) \leq \delta\}$ and $B^\circ_\delta(x) = \{y \in X : d(x, y) < \delta\}$ the *closed* and the *open balls* of radius δ and center x. A *δ-simplex* is a subset Y of X of diameter at most 2δ, i.e., $d(x, y) \leq 2\delta$ for any $x, y \in Y$.

Let S be a subset of X. For a given radius $\delta > 0$, a set of closed balls $\mathcal{C} = \{B_\delta(x_i) : i \in I\}$ with centers $x_i \in X$ is called a *covering* of S if $S \subseteq \bigcup_{i \in I} B_\delta(x_i)$. Analogously, a set of open balls $\mathcal{C}^\circ = \{B^\circ_\delta(x_i) : i \in I\}$ is called an *open covering* of S if $S \subseteq \bigcup_{i\in I} B^\circ_\delta(x_i)$. Denote by $\rho_\delta(S)$ (respectively, by $\rho^\circ_\delta(S)$) the minimum number of balls of radius δ in a covering (respectively, in a open covering) of S, and call $\rho_\delta(S)$ and $\rho^\circ_\delta(S)$ the *covering* and the *open covering numbers* of S. Obviously, $\rho_\delta(S) \le \rho^\circ_\delta(S)$. If S is compact, then $\rho^\circ_\delta(S)$ is finite, and therefore $\rho_\delta(S)$ is finite as well.

A set of closed balls $\mathcal{P} = \{B_\delta(x_i) : i \in I\}$ with centers $x_i \in S$ is called a *packing* of $S \subseteq X$ if the balls of $\mathcal{P}$ are pairwise disjoint. Analogously, a set of open balls $\mathcal{P}^\circ = \{B^\circ_\delta(x_i) : i \in I\}$ with centers $x_i \in S$ is called an *open packing* of S if the balls of $\mathcal{P}^\circ$ are pairwise disjoint. Denote by $\nu_\delta(S)$ the maximum number of closed balls in a packing of S, i.e., the size of a largest subset P of S such that $d(x_i, x_j) > 2\delta$ for any two distinct points x_i, x_j of P, and call $\nu_\delta(S)$ the *packing number* of S. Analogously, the *open packing number* $\nu^\circ_\delta(S)$ is the size of a largest subset P of S such that $d(x_i, x_j) \ge 2\delta$ for any two distinct points x_i, x_j of P. Clearly, for any $S \subseteq X$, the following inequalities hold: $\nu_\delta(S) \le \nu^\circ_\delta(S)$, $\nu_\delta(S) \le \rho_\delta(S)$, and $\nu^\circ_\delta(S) \le \rho^\circ_\delta(S)$. Therefore, if S is compact, then $\nu(S)$ and $\nu^\circ_\delta(S)$ are finite as well.

A subset $T \subseteq X$ is called a *hitting set* of $\mathcal{B}(S) = \{B_\delta(x) : x \in S\}$ if $T \cap B_\delta(x) \ne \varnothing$ for any $x \in S$. The *minimum hitting problem* asks to find a hitting set of $\mathcal{B}(S)$ of smallest cardinality $\tau_\delta(S)$. Notice that $\tau_\delta(S) = \rho_\delta(S)$ because if T is a hitting set of $\mathcal{B}(S)$, then the balls of radius δ centered at points of T cover the set S (the converse is also immediate). Therefore, further we will only speak about $\rho_\delta(S)$ and $\nu_\delta(S)$. Finally, an *δ-simplex covering* of S is a collection $\mathcal{R} = \{Y_i : i \in I\}$ of δ-simplices such that $Y_i \subseteq S$ and $S = \bigcup_{i\in I} Y_i$. The *δ-simplex covering number* $\theta_\delta(S)$ of S is the minimum number of δ-simplices in a covering of S. Notice that $\theta_\delta(S) = 1$ (i.e., S is an δ-simplex) if and only if $\nu_\delta(S) = 1$.

Kolmogorov and Tikhomirov [25] introduced these three covering and packing numbers (under different notations and names) and noticed the following simple but fundamental relationship between them: for any completely bounded (in particular, compact) subset S of an arbitrary metric space (X, d),

$$\nu_\delta(S) \le \theta_\delta(S) \le \rho_\delta(S) \le \nu_{\frac{\delta}{2}}(S).$$

Furthermore, they called the binary logarithms of the quantities $\theta_\delta(S), \rho_\delta(S)$, and $\nu_\delta(S)$ the *δ-entropy* of S, the *δ-entropy* of S with respect to X, and the *δ-capacity* of S, respectively (also called *metric entropy* and *metric capacity* of S). These quantities found numerous applications in pure and applied mathematics [27], probability theory and statistics [21], learning theory [26], and computational geometry [18], just to name some.

Notice also the following graph-theoretical interpretation of covering and packing numbers $\theta_\delta(S), \nu_\delta(S)$, and $\rho_\delta(S)$. For $\delta > 0$, the *Rips* (or the *Vietoris-Rips*) *complex* $P_\delta(S)$ of S [8, p.468] is a simplicial complex whose vertices are the points of S and a subset $Y \subseteq S$ is a simplex of $P_\delta(S)$ if and only if $\mathrm{diam}(Y) \le \delta$, i.e., if Y is a $\frac{\delta}{2}$-simplex. Denote by $G_\delta(S)$ the 1-skeleton of $P_\delta(S)$, i.e., S is the vertex-set of $G_\delta(S)$ and x, y are adjacent in $G_\delta(S)$ if and only if the pair x, y defines a simplex of $P_\delta(S)$, i.e., $d(x, y) \le \delta$. Notice that $P_\delta(S)$ is the clique complex of $G_\delta(S)$. Then a δ-simplex covering of S in the sense of Kolmogorov and Tikhomirov corresponds to a covering of S by simplices of $P_{2\delta}(S)$ and to a clique cover of $G_{2\delta}(S)$; therefore

$\theta_\delta(S)$ corresponds to the size of a minimum clique covering of $G_{2\delta}(S)$, i.e., to the chromatic number $\chi(\overline{G}_{2\delta}(S))$ of the complement $\overline{G}_{2\delta}(S)$ of the graph $G_{2\delta}(S)$. Analogously, a packing of S corresponds to a stable set of $G_{2\delta}(S)$, i.e., to a clique of $\overline{G}_{2\delta}(S)$; consequently, $\nu_\delta(S)$ equals the clique number $\omega(\overline{G}_{2\delta}(S))$ of the complement of $G_{2\delta}(S)$. Finally, $\rho_\delta(S)$ corresponds to the domination number of $G_\delta(S)$, i.e., to the minimum covering of S by stars of $G_\delta(S)$.

We will say that a class $\mathcal{M}$ of metric spaces has the *bounded covering-packing property* if there exists a universal constant c such that for any metric space (X,d) from $\mathcal{M}$, any $\delta > 0$, and any compact subset S of X, the inequality $\rho_\delta(S) \leq c\nu_\delta(S)$ holds. We will also say that $\mathcal{M}$ has the *bounded simplex-ball covering property*, if there exists a universal constant c such that for any $(X,d) \in \mathcal{M}$ and any $\delta > 0$, any δ-simplex S of X can be covered by at most c balls of radius δ. Recall also that a class $\mathcal{G}$ of graphs is *linearly χ-bounded* if there exists a constant c such that $\chi(G) \leq c\omega(G)$ for any graph $G \in \mathcal{G}$.

Lemma 1. *Let $\mathcal{M}$ be a class of metric spaces having the bounded simplex-ball covering property. If the class of graphs $\mathcal{G} = \{\overline{G}_{2\delta}(S) : \delta > 0$ and S is a compact subset of $X\}$ is linearly χ-bounded, then $\mathcal{M}$ satisfies the bounded covering-packing property.*

Proof. Since any coloring of $\overline{G}_{2\delta}(S)$ is a clique covering of $G_{2\delta}(S)$ and each clique of $G_{2\delta}(S)$ is a δ-simplex of S, the set S admits a δ-simplex covering with at most $c\omega(\overline{G}_{2\delta}(S))$ simplices. If (X,d) has the bounded covering-packing property with constant c', we conclude that S can be covered with at most $c'c\omega(\overline{G}_{2\delta}(S)) = c'c\nu_\delta(S)$ balls of radius δ. □

An important class of metric spaces satisfying the bounded covering-packing property (and extending the Euclidean spaces) is constituted by metric spaces with *bounded doubling dimension*, i.e., metric spaces (X,d) in which for any $\delta > 0$ any ball of radius 2δ of X can be covered with a constant number of balls of radius δ [18]. We will relax this doubling property in the following way. We will say that a metric space (X,d) satisfies the *weak doubling property* if there exists a constant c such that for any $\delta > 0$ and any compact set $S \subseteq X$, there exists a point $v \in S$ such that $B_{2\delta}(v) \cap S$ can be covered with at most c balls of radius δ of X. The proof of the following result will be given in the next section:

Proposition 1. *If a complete metric space (X,d) satisfies the weak doubling property with constant c, then for any compact set $S \subseteq X$ and any $\delta > 0$, $\rho_\delta(S) \leq c\nu_\delta(S)$.*

2.2. **Related work.** It was shown in [16] that the class $\mathcal{M}_{\mathsf{planar}}$ of all metric spaces obtained as standard graph-metrics of planar graphs has the bounded simplex-ball covering property. In [6], this result was generalized to all graphs on surfaces of a given genus; see also [10, 11] for other generalizations of the result of [16]. It was conjectured in [12, Problem 5] that the class $\mathcal{M}_{\mathsf{planar}}$ has the bounded covering-packing property, namely, that it satisfies the weak doubling property. Notice also, that it was shown in [15] that if S is a compact subset of a geodesic ϵ-hyperbolic space (in the sense of Gromov) or of an ϵ-hyperbolic graph, then $\rho_{\delta+2\epsilon}(S) \leq \nu_\delta(S)$ (compare it with the general inequality $\nu_\delta(S) \leq \rho_\delta(S) \leq \nu_{\frac{\delta}{2}}(S)$). This result can be interesting if the hyperbolicity ϵ constant is much smaller than the radius δ of balls used in the covering.

There exists a strong analogy between the properties of graphs and geodesic metric spaces, due to their uniform local structure. Any graph $G = (V, E)$ gives rise to a network-like geodesic space (into which G isometrically embeds) obtained by replacing each edge xy of G by a segment isometric to $[0, 1]$ with ends at x and y. Conversely, by [8, Proposition 8.45], any geodesic metric space (X, d) is (3,1)-quasi-isometric to a graph $G = (V, E)$. This graph G is constructed in the following way: let V be an open $\frac{1}{3}$-packing of X (it exists by Zorn's lemma but can be infinite). Then two points $x, y \in V$ are adjacent in G if and only if $d(x, y) \leq 1$.

Due to this analogy, one can formulate the previous question about $\mathcal{M}_{\mathsf{planar}}$ for their continuous counterparts $\mathcal{M}_{\mathsf{polygon}}$— polygons in $\mathbb{R}^2$ endowed with the (intrinsic) geodesic metric. It turns out that this question was not yet considered even for simple polygons (in this case, only a factor 2 approximation algorithm for packing number was recently given in [35]). The geodesic metric on simple polygons was studied in several papers in connection with algorithmic problems. In particular, in was shown in [31], that balls are convex, implying that simple polygons are Busemann spaces. In this paper, we consider the relationship between the packing and covering numbers not only for simple polygons in the Euclidean or hyperbolic planes but also for (compact subsets of) general Busemann surfaces.

2.3. **Geodesics and geodesic metric spaces.** Let (X, d) be a metric space. A *path* in X is a continuous map $\gamma : [a, b] \to X$, where a and b are two real numbers with $a \leq b$. If $\gamma(a) = x$ and $\gamma(b) = y$, then x and y are the *endpoints* of γ and that γ joins x and y. A *geodesic* in X is a path $\gamma : [a, b] \to X$ that is distance-preserving, that is, such that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. A *geodesic line* (or simply a *line*) is a distance-preserving map $\gamma : \mathbb{R} \to X$ and a *geodesic ray* (or simply a *ray*) is a distance-preserving map $\gamma : [0, \infty) \to X$. A path $\gamma : [a, b] \to X$ is said to be a *local geodesic* if for all $a < t < b$ one can find a closed interval $I(t) \subseteq [a, b]$ containing t in its interior such that the restriction of γ on $I(t)$ is geodesic. A metric space X is *geodesic* if every pair of points in X can be joined by a geodesic. A *uniquely geodesic space* is a geodesic space in which every pair of points can be joined by a unique geodesic.

2.4. **Busemann surfaces.** A *planar surface* (without boundary) $\mathcal{S}$ is a 2-dimensional manifold homeomorphic to the plane $\mathbb{R}^2$. A geodesic metric space $(\mathcal{S}, d)$ is called a *Busemann surface* if $\mathcal{S}$ is a planar surface and the metric space $(\mathcal{S}, d)$ is a Busemann space. A *Busemann space* (or a *non-positively curved space in the sense of Busemann*) is a geodesic metric space (X, d) in which the distance function between any two geodesics is convex: for any two reparametrized geodesics $\gamma : [a, b] \to X$ and $\gamma' : [a', b'] \to X$ the map $f_{\gamma,\gamma'}(t) : [0, 1] \to \mathbb{R}$ defined by $f_{\gamma,\gamma'}(t) = d(\gamma((1 - t)a + tb), \gamma'((1 - t)a' + tb'))$ is convex.

Busemann spaces satisfy many fundamental metric, geometric, and topological properties: they are contractible, have the fixed point property, are uniquely geodesic, local geodesics are geodesics, open and closed balls are convex, projections on convex sets are unique, and geodesics vary continuously with their endpoints. They can be characterized in a pretty local-to-global way: every complete geodesic locally compact, locally convex and simply connected metric space is a Busemann space. For these and other results on Busemann spaces consult the book of Papadopoulos [30].

Basic examples of Busemann spaces are the Euclidean space $\mathbb{E}^n$, and more generally, normed strictly convex vector spaces, the hyperbolic n-dimensional space $\mathbb{H}^n$, $\mathbb{R}$-trees, and Riemannian manifolds of global nonpositive sectional curvature. A large subclass of Busemann spaces is constituted by non-positively curved spaces in the sense of Alexandrov, known also under the name of CAT(0) spaces [8].

Our motivating examples of Busemann surfaces are the simple polygons P in the plane endowed with the intrinsic geodesic metric. After triangulating P, one can view P as a finite 2-dimensional cell complex and, as noticed in [13, Subsection 2.4], P can be extended to a Busemann surface $\mathcal{S}$ so that P will be a convex subset of $\mathcal{S}$.

2.5. **The main results.** We continue with the formulation of the main results of this note. Starting from now, we will denote $\rho_\delta(S)$ and $\nu_\delta(S)$ by $\rho(S)$ and $\nu(S)$, respectively.

Theorem 1. *Let S be a compact subset of a Busemann surface $(\mathcal{S}, d)$ and δ an arbitrary positive number. Then $\rho(S) \leq 19\nu(S)$.*

Corollary 1. *Let $\mathcal{P}$ be a simple polygon in $\mathbb{R}^2$. Then $\nu(\mathcal{P}) \leq \rho(\mathcal{P}) \leq 19\nu(\mathcal{P})$ for any $\delta > 0$.*

The proof of Theorem 1 immediately follows from Proposition 1 and two other propositions formulated below. The first proposition extends the well-known folkloric result by Hadwiger and Debrunner [22] that any set of pairwise intersecting unit balls in the plane can be pierced by three needles. Namely, it shows that Busemann surfaces satisfy the bounded simplex-ball covering property with constant 3:

Proposition 2. *Let S be a compact subset of a Busemann surface $(\mathcal{S}, d)$ and suppose that the diameter of S is at most 2δ. Then S can be covered with 3 balls of radius δ, i.e., $\rho(S) \leq 3$.*

The second result shows that Busemann surfaces satisfy the weak doubling property:

Proposition 3. *Let S be a compact subset of a Busemann surface $(\mathcal{S}, d)$ and let $u, v \in S$ be a diametral pair of S. Then $B_{2\delta}(v) \cap S$ can be covered by 19 balls of radius δ.*

The idea of proof of Proposition 3 is to partition the set $B_{2\delta}(v) \cap S$ into six regions, four of them of diameter $\leq 2\delta$ and to which we can apply Proposition 2 and two regions which can be covered with eight balls.

Remark 1. Notice that in general Busemann surfaces (at the difference of Euclidean and hyperbolic planes) do not satisfy the doubling property. Indeed, for any positive integer n, the star with n leaves, center v, and length 2δ of all edges can be embedded isometrically into a Busemann surface. Since the distance between any two leaves is 4δ, any covering of the ball $B_{2\delta}(v)$ with balls of radius δ requires at least n balls.

3. Proofs

In this section, we provide the proof of Propositions 1-3 and Corollary 1. For this, we will need some auxiliary geometric properties of Busemann surfaces, which we present next.

3.1. **Auxiliary results.** In this subsection, we present elementary properties of Busemann planar surfaces $(\mathcal{S}, d)$. For two points $x, y \in \mathcal{S}$, we denote by $[x, y]$ the unique geodesic segment joining x and y. A set $R \subseteq \mathcal{S}$ is called *convex* if $[p, q] \subseteq R$ for any $p, q \in R$. For a set Q of $\mathcal{S}$ the smallest convex set $\operatorname{conv}(Q)$ containing Q is called the *convex hull* of Q.

A geodesic metric space (X, d) is said to have the *geodesic extension property* if the geodesic $[x, y]$ between any two distinct points x, y can be extended to a *geodesic line*, i.e., to a line (x, y) passing via x and y. Based on [8, Footnote 24], it was noticed in [13, Lemma 1] that Busemann surfaces have the extension property:

Lemma 2. *$\mathcal{S}$ has the geodesic extension property.*

For a geodesic line ℓ, we denote by H'_ℓ and H''_ℓ the unions of the two connected components of $\mathcal{S} \setminus \ell$ with ℓ. We call H'_ℓ and H''_ℓ *closed halfplanes.* Since each line is convex, H'_ℓ and H''_ℓ are convex sets of $\mathcal{S}$. We will say that a line ℓ *separates* two sets A and B if A and B belong to different closed halfplanes defined by ℓ.

We continue by recalling the following fundamental properties of Busemann surfaces, which in fact characterize the Busemann spaces among geodesic metric spaces:

Lemma 3. [30, Proposition 8.1.2(v)&(vi)] *(i) Let $[x_0, x_1]$ and $[x'_0, x'_1]$ be two geodesics of $\mathcal{S}$ and let m and m' be their respective midpoints. Then $d(m, m') \leq \frac{1}{2}(d(x_0, x'_0) + d(x_1, x'_1))$.*

(ii) Let $[x_0, x_1]$ and $[x_0, x'_1]$ be two arbitrary geodesics of $\mathcal{S}$ having a common initial point x_0. For all $t \in [0, 1]$, let x_t and x'_t be the points of $[x_0, x_1]$ and $[x_0, x'_1]$, respectively, and satisfying $d(x_0, x_t) = t \cdot d(x_0, x_1)$ and $d(x_0, x'_t) = t \cdot d(x_0, x'_1)$. Then $d(x_t, x'_t) \leq t \cdot d(x_1, x'_1)$.

Lemma 4. [30, Corollary 8.2.3] *Every local geodesic of $\mathcal{S}$ is a geodesic.*

The next lemma immediately follows from the definition of Busemann spaces.

Lemma 5. *Closed balls $B_r(x)$ of $\mathcal{S}$ are convex.*

For three points x, y, z of $\mathcal{S}$, the *geodesic triangle* $\Delta(x, y, z)$ is the union of the three geodesics $[x, y], [y, z]$, and $[z, x]$. We will call the closed region $\Delta^*(x, y, z)$ of $\mathcal{S}$ bounded by $\Delta(x, y, z)$ a *triangle* with vertices x, y, z. We will say that the triangle $\Delta^*(x, y, z)$ is *degenerated* if the points x, y, z are collinear, i.e., one of these points belongs to the geodesic between the other two. Analogously, by a *(convex) quadrangle* we will mean the convex hull of four point x, y, z, v in convex position, i.e., neither of the four points is in the convex hull of the other three. For two distinct points $x, y \in \mathcal{S}$, let $C(x, y) = \{z \in \mathcal{S} : x \in [y, z]\}$ and $C(y, x) = \{z \in \mathcal{S} : y \in [x, z]\}$; we will call the sets $C(x, y)$ and $C(y, x)$ *cones.* Since $\mathcal{S}$ satisfies the geodesic extension property, the set $C(x, y) \cup [x, y] \cup C(y, x)$ can be equivalently defined as the union of all geodesic lines extending $[x, y]$.

We continue by recalling some results from [13]. We start with a Pasch axiom, which we formulate in a slightly stronger but equivalent form:

Lemma 6. [13, Lemma 6] *(Pasch axiom) If $\Delta(x, y, z)$ is a geodesic triangle, $u \in [x, y], v \in [x, z]$, and $p \in [y, z]$, then $[u, v] \cap [x, p] \neq \varnothing$.*

Lemma 7. [13, Lemma 7] *The cones $C(x,y)$ and $C(y,x)$ are convex and closed subsets of $\mathcal{S}$.*

Lemma 8. [13, Lemma 8] *$\Delta^*(x,y,z)$ coincides with the convex hull of x,y,z.*

Lemma 9. [13, Lemma 9] *(Peano axiom) If $\Delta(x,y,z)$ is a geodesic triangle, $p \in [x,y]$, $q \in [x,z]$, and $u \in [p,q]$, then there exists a point $v \in [y,z]$ such that $u \in [x,v]$.*

Since a Busemann surface $\mathcal{S}$ is homeomorphic to the plane $\mathbb{R}^2$, the properties of $\mathbb{R}^2$ preserved by homeomorphisms also hold in $\mathcal{S}$. For example, any simple closed curve γ in $\mathcal{S}$ divides the surface $\mathcal{S}$ into an interior region $\mathcal{R} := \mathcal{R}(\gamma)$ bounded by γ and an exterior region. Moreover, $\mathcal{R}$ is a contractible bounded subset of $\mathcal{S}$. A *cut* of $\mathcal{R}$ with endpoints $x, y \in \gamma$ is a path $\mu : [a,b] \to \mathcal{R}$ such that $\mu(a) = x, \mu(b) = y$, and $\mu(c) \in \mathcal{R}$ for any $a \le c \le b$. Using the homeomorphism between $\mathcal{S}$ and $\mathbb{R}^2$, one can see that any cut μ of $\mathcal{R}$ divides $\mathcal{R}$ into two contractible bounded regions. Analogously, if x, u, y, v are four points occurring in this order on γ, μ' is a cut of $\mathcal{R}$ with endpoints x, y, and μ'' is a cut of $\mathcal{R}$ with endpoints u, v, then μ' and μ'' cross and divide $\mathcal{R}$ into four contractible regions.

We will use this kind of results to derive the following basic properties of Busemann surfaces:

(1) If $\Delta^*(x,y,z)$ is a triangle and $t \in [y,z]$, then $\Delta^*(x,y,z)$ is divided into two triangles $\Delta^*(x,y,t)$ and $\Delta^*(x,z,t)$ (i.e., $\Delta^*(x,y,z) = \Delta^*(x,y,t) \cup \Delta^*(x,t,z)$ and $\Delta^*(x,y,t) \cap \Delta^*(x,t,z) = [x,t]$);

(2) If $\Delta^*(x,y,z)$ is a triangle and $u \in [x,y], v \in [x,z]$, and $w \in [y,z]$, then $\Delta^*(x,y,z)$ is divided into four triangles $\Delta^*(x,u,v), \Delta^*(v,w,z), \Delta^*(u,w,y)$, and $\Delta^*(u,v,w)$;

(3) If $\Delta^*(x,y,z)$ is a triangle and $u \in \Delta^*(x,y,z)$, then $\Delta^*(x,y,z)$ is divided into three triangles $\Delta^*(x,y,u), \Delta^*(y,z,u)$, and $\Delta^*(x,z,u)$;

(4) If $Q = \operatorname{conv}(x,y,z,u)$ is a convex quadrangle with sides $[x,y],[y,z],[z,u],[u,x]$ and $p \in [x,y], s \in [y,z], q \in [z,u], t \in [u,x]$, then the geodesic segments $[p,q]$ and $[s,t]$ divide Q into four convex quadrangles.

We will denote by $\partial B_r(x)$ the *sphere* of center x and radius r; $\partial B_r(x)$ can be viewed as the difference between $B_r(x)$ and $B_r^\circ(x)$ or, equivalently, as the set $\{y \in \mathcal{S} : d(x,y) = r\}$. The following property is also a consequence of the homeomorphism between $\mathcal{S}$ and $\mathbb{R}^2$:

Lemma 10. *Any sphere $\partial B_r(x)$ of $\mathcal{S}$ is homeomorphic to the circle $\mathbb{S}^1$ of $\mathbb{R}^2$.*

We continue with some new properties of Busemann surfaces. Let $\pi(x,y,z)$ denote the perimeter of $\Delta^*(x,y,z)$, i.e., $\pi(x,y,z) = d(x,y) + d(y,z) + d(z,x)$. Then the following monotonicity properties of triangles holds:

Lemma 11. *If $x', y', z' \in \Delta^*(x,y,z)$, then $\pi(x',y',z') \le \pi(x,y,z)$. Moreover, the equality holds only if either $\{x',y',z'\} = \{x,y,z\}$ or $\Delta^*(x,y,z)$ is degenerated, i.e., the points x,y,z are collinear.*

Proof. We proceed by induction on the number $k := k(x',y',z')$ of points x', y', z' not belonging to $\Delta(x,y,z)$. If $k = 0$, then $x', y', z' \in \Delta(x,y,z)$ and the inequality $\pi(x',y',z') \le \pi(x,y,z)$

easily follows by applying the triangle inequality. Indeed, if for example $x' \in [y,z], y' \in [x,z]$, and $z' \in [x,y]$, then by triangle inequality $d(x',y') \le d(x',z) + d(z,y')$, $d(y',z') \le d(y',x) + d(x,z')$, and $d(z',x') \le d(z',y)+d(y,x')$. Consequently, $\pi(x',y',z') \le \pi(x,y,z)$. Analogously, if $x' \in [y,z]$ and $y',z' \in [x,z]$ with $z' \in [y',z]$, then $d(y',x') \le d(y',x) + d(x,y) + d(y,x')$, $d(x',z') \le d(x',z) + d(z,z')$ and again $\pi(x',y',z') \le \pi(x,y,z)$.

Therefore, suppose that $k > 0$, namely that $y' \notin \Delta(x,y,z)$. Consider a geodesic extension of $[x',y']$ in the direction of y'. Then we will find a point $y'' \in \Delta(x,y,z)$ such that $y' \in [x',y'']$. Since $k(x',y'',z') = k(x',y',z') - 1$, by induction assumption $\pi(x',y'',z') \le \pi(x,y,z)$. On the other hand, $y' \in [x',y'']$, hence applying the basic case $k = 0$ to $\Delta(x',y'',z'')$ and x',y',z', we conclude that $\pi(x',y',z') \le \pi(x',y'',z')$. Consequently, $\pi(x',y',z') \le \pi(x,y,z)$.

Now, consider the case of equality between $\pi(x',y',z')$ and $\pi(x,y,z)$. Since in previous proof we reduced each case with $k > 0$ to the case $k-1$ using inequalities, it suffices to consider the basic case $k = 0$. Let $x' \in [y,z], y' \in [x,z]$, and $z' \in [x,y]$ and suppose that $x' \ne z$ and $y' \ne x, z$ (the cases $x' \in [y,z], z', y' \in [x,z]$ and $y' = x, z' = y$, $x' \in [y,z] \setminus \{y,z\}$ are similar). Since $\pi(x',y',z') = \pi(x,y,z)$, the triangle inequality $d(x',y') \le d(x',z) + d(z,y')$ must be an equality, i.e., $z \in [x',y']$. But then the path obtained by concatenating the geodesics $[x,z]$ and $[z,y]$ along z is a local geodesic. By Lemma 4 this path is a geodesic, thus $z \in [x,y]$ and the triangle $\Delta^*(x,y,z)$ is degenerated. □

Lemma 12. *If $u,v \in \Delta^*(x,y,z)$ and $d(x,y), d(y,z), d(z,x) \le \delta$, then $d(u,v) \le \delta$.*

Proof. First notice that each of the points x,y,z is at distance at most δ from all points of $\Delta^*(x,y,z)$: indeed, since $x,y,z \in B_\delta(x)$ and the ball $B_\delta(x)$ is convex, $\Delta^*(x,y,z) \subseteq B_\delta(x)$. Hence $x,y,z \in B_\delta(u)$. Again, since $B_\delta(u)$ is convex, $v \in \Delta^*(x,y,z) \subseteq B_\delta(u)$, whence $d(u,v) \le \delta$. □

We continue with the following quadrangle condition:

Lemma 13. *If x,y,u,v are four points of $\mathcal{S}$ such that $[x,y] \cap [u,v] \ne \varnothing$, then* $\max\{d(x,u) + d(y,v), d(x,v) + d(y,u)\} \le d(x,y) + d(u,v)$.

Proof. Let $z \in [x,y] \cap [u,v]$. By triangle inequality, $d(x,u) \le d(x,z) + d(z,u)$ and $d(v,y) \le d(v,z)+d(z,y)$. Hence, $d(x,u)+d(v,y) \le d(x,z)+d(z,u)+d(v,z)+d(z,y) = d(x,y)+d(u,v)$. The case of equality is straightforward. □

The following lemma is a very particular case of a result of [24] established for all n-dimensional uniquely geodesic spaces.

Lemma 14. *(Helly number) Any collection $\mathcal{C} = \{C_i : i \in I\}$ of compact convex sets of $\mathcal{S}$ has a nonempty intersection provided any three sets of $\mathcal{C}$ have a nonempty intersection. In particular, any collection of closed balls $\mathcal{B}$ of $\mathcal{S}$ has a nonempty intersection provided any three balls of $\mathcal{B}$ intersect.*

Proof. Since the sets of $\mathcal{C}$ are compact, it suffices to establish this Helly property for finite collections. By the definition of the Helly number [33], it suffices to show that $\mathcal{S}$ does not contain a quadruplet of points $X = \{x_1,x_2,x_3,x_4\}$ such that $\bigcap_{i=1}^4 \mathrm{conv}(X \setminus \{x_i\}) = \varnothing$.

Indeed, pick any quadruplet $X = \{x_1, x_2, x_3, x_4\}$ of $\mathcal{S}$ and suppose that $x_i \notin \text{conv}(X \setminus \{x_i\})$ for $i = 1, \ldots, 4$. Since $\text{conv}(x_1, x_2, x_3) = \Delta^*(x_1, x_2, x_3)$ by Lemma 8, $x_4 \notin \Delta^*(x_1, x_2, x_3)$. If $[x_1, x_4] \cup [x_2, x_4] \cup [x_3, x_4]$ does not intersect $\Delta(x_1, x_2, x_3)$ in other points than x_1, x_2, x_3, then, since $\mathcal{S}$ is homeomorphic to $\mathbb{S}^1$ (Lemma 10), necessarily one of the points x_1, x_2, x_3, say x_3, will belong to the region bounded by the geodesics between x_1, x_2, and x_4. But in this case, $x_3 \in \Delta^*(x_1, x_2, x_4) =\text{conv}(x_1, x_2, x_4)$, contrary to our choice of X. Thus the geodesic between x_4 and one of the points x_1, x_2, x_3, say x_1, intersects the opposite side $[x_2, x_3]$ of $\Delta(x_1, x_2, x_3)$. Let $y \in [x_1, x_4] \cap [x_2, x_3]$. But then $y \in \bigcap_{i=1}^4 \text{conv}(X \setminus \{x_i\})$, and we are done. The second assertion for balls immediately follows, since the closed balls are convex. □

For a compact set S and a point $u \in S$, the *eccentricity* of u in S is $e_S(u) = \max\{d(u, v) : v \in S\}$. The *diameter* $\text{diam}(S)$ of S is the maximum eccentricity of a point u of S, i.e., $\text{diam}(S) = \max\{d(u, v) : u, v \in S\}$.

Lemma 15. *For any compact set S of $\mathcal{S}$, any point $u \in S$ has the same eccentricity in the sets conv(S) and S. Moreover, the sets S and conv(S) have the same diameter.*

Proof. Let $r := e_S(u)$ and $R := \text{diam}(S)$. The set $\text{conv}(S)$ can be constructed as the directed union of the sets $S_0 = S \subseteq S_1 \subseteq S_2 \subseteq \ldots$, where $S_i = \bigcup_{x,y \in S_{i-1}} [x, y]$. By induction on i we will prove that $e_{S_i}(u) = r$ and $\text{diam}(S_i) = R$. This is obvious for $i = 0$. Suppose this holds for all $j < i$ and pick any two points $x, y \in S_i$. By definition, there exist four points $x', x'', y', y'' \in S_{i-1}$ such that $x \in [x', x'']$ and $y \in [y', y'']$. Since $\text{diam}\{x', x'', y', y''\} \leq R$, $x', x'' \in B_R(y') \cap B_R(y'')$. By convexity of balls, $x \in B_R(y') \cap B_R(y'')$, i.e., $d(x, y'), d(x, y'') \leq R$. Hence $y', y'' \in B_R(x)$. Consequently, since $y \in [y', y'']$ and $B_R(x)$ is convex, $d(x, y) \leq R$, i.e., $\text{diam}(S_i) = R$. Analogously, since $d(u, x'), d(u, x'') \leq r$, the convexity of the ball $B_r(u)$ implies that $d(u, x) \leq r$, whence $e_{S_i}(u) = r$. □

For a point x and a geodesic segment $[y, z]$, the *shadow* of $[y, z]$ with respect to x is the set

$$C(x, [y, z]) := \{v \in \mathcal{S} : \text{ some line } (x, v) \text{ extending } [x, v] \text{ separates } y \text{ from } z\}.$$

The *shadow* $C(x, \Delta^*(u, y, z))$ of a triangle $\Delta^*(u, y, z)$ with respect to a point $x \notin \Delta^*(u, y, z)$ is the union of the shadows of its three sides with respect to x.

Lemma 16. *For any point x, any geodesic segment $[y, z]$, and any triangle $\Delta^*(u, y, z)$ not containing x, the shadows $C(x, [y, z])$ and $C(x, \Delta^*(u, y, z))$ are convex.*

Proof. Let $p, q \in C(x, [y, z])$ and $t \in [p, q]$. Let (x, p) be a line passing via x and p and separating y from z. Analogously, let (x, q) be a line passing via x and q and separating y from z. Let $p' \in [y, z] \cap (x, p)$ and $q' \in [y, z] \cap (x, q)$. Suppose without loss of generality that y, p', q', z occur in this order on $[y, z]$. By Pasch axiom there exists a point $s' \in [p', q'] \cap [x, s]$. Since $s' \in [y, z]$, any line (x, s) extending $[x, s]$ separates p' from q' and y from z. Consequently, $s \in C(x, [y, z])$, establishing the convexity of $C(x, [y, z])$.

To prove the convexity of $C(x, \Delta^*(u, y, z))$ it suffices to notice that for any line ℓ, the intersections of ℓ with the shadows $C(x, [y, z]), C(x, [u, y])$, and $C(x, [u, z])$ are three pairwise intersecting segments of ℓ; thus their union is also a segment of ℓ. □

Lemma 17. *If $v \notin \Delta^*(x,y,z)$, then there exists a line ℓ extending one side of $\Delta^*(x,y,z)$ and separating v and $\Delta^*(x,y,z)$.*

Proof. First suppose that there exists a point $p \in \Delta^*(x,y,z) \setminus \Delta(x,y,z)$. Suppose that $[v,p]$ intersects a side of $\Delta^*(x,y,z)$, say $[x,y]$ in a point q and $[v,q] \cap \Delta^*(x,y,z) = \{q\}$. First suppose that $q \neq x,y$. Let (x,y) be a line extending $[x,y]$. Then v and p belong to different closed halfplanes defined by (x,y). Since $\Delta^*(x,y,z)$ belongs to the halfplane of p, we can take (x,y) as the separating line ℓ. Now suppose that $q = x$. Let (y,x) and (z,x) be two lines extending $[y,x]$ and $[z,x]$, respectively. Let r_y and r_z be the rays of (y,x) and (z,x) with origin at x and not passing via the points y and z, respectively. If one of the lines $(y,x),(z,x)$ separates v from $\Delta^*(x,y,z)$, then we are done. Otherwise, we assert that the union ℓ' of the ray r_z with the ray $[x,y)$ with origin x and passing via y is a line. For this is suffices to show that ℓ' is locally convex in the neighborhood of x. Suppose not; then we can find a point $y' \in [x,y]$ and a point $z' \in r_z$ such that $y',z' \neq x$ and $[y',z'] \cap \ell' = \{y',z'\}$. Then either $[y',z']$ intersects r_y in a point y'' or $[y',z']$ intersects $[x,z]$ in a point z''. In both cases we conclude that y' and y'' or z'' and z' are connected by two geodesics. This establishes that ℓ' is a line. Since ℓ' separates v from $\Delta^*(x,y,z)$, we are done. (Notice that it may happen that the rays r_y and r_z coincide with the ray $[x,v)$ with origin x and passing via v. In this case ℓ' separates – but non strictly– v from $\Delta^*(x,y,z)$).

Finally suppose that $\Delta^*(x,y,z) = \Delta(x,y,z)$. If x,y and z are collinear, then as ℓ one can take any line passing via x,y, and z. Otherwise, one can easily see that $\Delta(x,y,z)$ is a tripod, i.e., there exists a point $w \neq x,y,z$ such that $\Delta(x,y,z) = [x,w] \cup [y,w] \cup [z,w]$. The unions of any pairs of rays $[w,x),[w,y)$, and $[w,z)$ with origin w and passing via respectively x,y, and z are lines. Each of these three lines define a closed halfplane not containing the third vertex of $\Delta^*(x,y,z)$. The union of these three closed halfplanes is the surface $\mathcal{S}$ and the halfplanes pairwise intersect in the rays r_x,r_y, and r_z. Then as ℓ we can take that of the three lines which defines the closed halfplane containing v. □

3.2. **Proof of Proposition 1.** The proof of Proposition 1 is algorithmic and builds simultaneously (in a primal-dual way) a covering $\mathcal{C}$ of S with closed δ-balls and an open packing P of S satisfying the inequality $|\mathcal{C}| \leq c|P|$. Since P is an open packing and S is compact, $|P| \leq \nu^\circ(S) \leq \rho^\circ(S) < \infty$, thus P and $\mathcal{C}$ are finite and their construction requires a finite number of steps. Then using local perturbations, we will show how to transform P into a packing P' of the same size as P.

Start by setting $S_0^* := S$, $S_0 := S$, $\mathcal{C} := \varnothing$, $P := \varnothing$, and $i = 0$. While $S_i \neq \varnothing$, set $S_i^* := \overline{S}_i$ (the closure of S_i). Since (X,d) is complete, S_i^* is compact. Since (X,d) satisfies the weak doubling property, S_i^* contains a point v such that the set $B_{2\delta}(v) \cap S_i^*$ can be covered with $k \leq c$ balls $B_\delta(x_1),\ldots,B_\delta(x_k)$ of radius δ of X. Add the balls $B_\delta(x_1),\ldots,B_\delta(x_k)$ to the covering $\mathcal{C}$, denote the point v by p_i and add it to P. Finally, set $S_{i+1} := S_i \setminus (\bigcup_{j=1}^k B_\delta(x_j))$ and $S_{i+1}^* := \overline{S}_{i+1}$, and apply the algorithm to these two new sets.

We claim that P is an open packing of S. Pick any pair of points $p_i,p_j \in P$ and let $j < i$. Then p_i is either a point of S_i or p_i is the limit of an infinite sequence $\{s_t\}$ of points of S_i.

From its definition, the set S_i consists of all yet not covered by $\mathcal{C}$ points of S; in particular, we have $S_i \cap (\bigcup_{k=1}^{i-1} B_{2\delta}(p_k)) = \varnothing$. Consequently, if $p_i \in S_i$, since $p_i \notin B_{2\delta}(p_j)$, we conclude that $d(p_i, p_j) > 2\delta$ in this case. Now, suppose that p_i is the limit of a sequence $\{s_t\}$ of points of S_i. If $d(p_i, p_j) < 2\delta$, then for any $\epsilon > 0$ such that $d(p_i, p_j) + \epsilon < 2\delta$, all points of $\{s_t\}$ except a finite number will be in the ϵ-neighborhood of p_i. For any such point s_t, we will have $d(s_t, p_j) \leq d(s_t, p_i) + d(p_i, p_j) \leq \epsilon + d(p_i, p_j) < 2\delta$, contrary to the choice of s_t from S_i. This contradiction shows that P is an open packing of S. Consequently, P and $\mathcal{C}$ are finite, and from their construction, $|\mathcal{C}| \leq c|P|$.

Now, we will show how to transform the finite open packing $P = \{p_1, \ldots, p_n\}$ of S into a packing P' of the same size. For this we will move each point of P at most once. We proceed the points of P in the reverse order and for each point p_i of P either we include it in P' (and denote it by p'_i) or include in P' a point $p'_i \in S_i$. Suppose that after proceeding the points $p_n, \ldots, p_{i+1}$, the set P' has the form $P' = \{p_1, \ldots, p_i, p'_{i+1}, \ldots, p'_n\}$ and satisfies the following invariants: (a) $d(p_j, p'_k) > 2\delta$ for any $j = 1, \ldots, i$ and $k = i+1, \ldots, n$ and (b) $d(p'_j, p'_k) > 2\delta$ for any $i+1 \leq j < k \leq n$. We will show how to proceed the point p_i to keep valid the invariants (a) and (b). If $d(p_i, p_j) > 2\delta$ for any $j < i$, then we simply set $p'_i = p_i$ and obviously (a) and (b) are preserved. Otherwise, suppose that there exists a point p_j with $j < i$ such that $d(p_i, p_j) = 2\delta$. By the construction of P and the argument in the proof that P is an open packing, we conclude that $p_i \notin S_i$ and therefore p_i is a limit of an infinite sequence $\{s_t\}$ of points of S_i. In the basis case $i = n$ we simply pick as p'_n any point from the sequence $\{s_t\}$. Obviously, the conditions (a) and (b) will be preserved. Now, suppose that $i < n$. Let $\epsilon := \min\{d(p_i, p'_k) - 2\delta : k > i\}$. Clearly, $\epsilon > 0$. Pick as p'_i any point of the sequence $\{s_t\}$ lying in the $\frac{\epsilon}{2}$-neighborhood of p_i. Then $d(p_j, p'_i) > 2\delta$ for any $j < i$, because $p'_i \in S_i$. Also $d(p'_i, p'_k) > 2\delta$ for any $k > i$ because by triangle inequality $d(p'_i, p'_k) > d(p_i, p'_k) - d(p'_i, p_i) > d(p_i, p'_k) - \frac{\epsilon}{2} > 2\delta$. This shows that after proceeding all points of P, we will obtain a set P' of n points of S, satisfying the conditions (a) and (b), i.e., a packing of S. This finishes the proof of Proposition 1.

3.3. **Proof of Proposition 2.** Let S be a compact subset of $(\mathcal{S}, d)$ and suppose that the diameter of S is at most 2δ. Since by Lemma 15, the diameter of $\text{conv}(S)$ coincides with the diameter of S and $\text{conv}(S)$ is compact, we will further assume without loss of generality that S is convex. We will prove that S can be covered three balls of radius δ. Since $\text{diam}(S) \leq 2\delta$, any two balls centered at points of S intersect. If any three such balls intersect, then Lemma 14 implies that $\bigcap_{x \in S} B_\delta(x) \neq \varnothing$ and if v is an arbitrary point from this intersection, then $S \subseteq B_\delta(v)$. Therefore, further we can suppose that S contains triplets of points such that the δ-balls centered at these points have an empty intersection. We will call such triplets *critical*.

Let $x, y, z \in S$ be an arbitrary triplet of points of S. Denote by x^*, y^*, and z^* the midpoints of the geodesics $[y, z], [x, z]$, and $[x, y]$, respectively. Since $d(x, y), d(y, z), d(z, x) \leq 2\delta$, from Lemma 3 we conclude that $d(x^*, y^*), d(y^*, z^*), d(z^*, x^*) \leq \delta$. Let $A_x := \Delta^*(x, y, z) \cap B_\delta(y) \cap B_\delta(z)$, $A_y := \Delta^*(x, y, z) \cap B_\delta(x) \cap B_\delta(z)$, and $A_z := \Delta^*(x, y, z) \cap B_\delta(x) \cap B_\delta(y)$. These sets are compact (as the intersection of compact sets) and nonempty (because $x^* \in A_x, y^* \in A_y$,

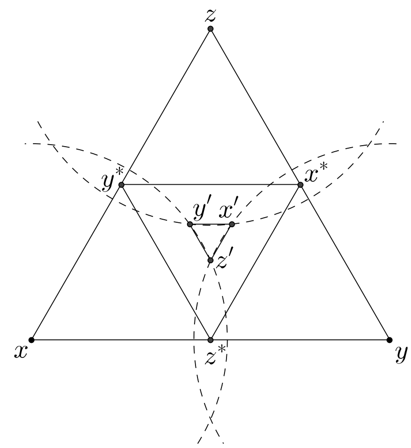

FIGURE 1. To the proof of Proposition 2.

and $z^* \in A_z$). Among all triplets of points, one from each of the sets A_x, A_y, and A_z, let x', y', z' be a triplet with the minimum perimeter $\pi(x', y', z')$ of $\Delta^*(x', y', z')$. Such a triplet exists because the sets A_x, A_y, and A_z are compact. If the triplet x, y, z is not critical, then the points x', y', z' coincide. We will call $\Delta^*(x', y', z')$ a *critical triangle* for the triplet x, y, z. We continue with simple properties of critical triplets and their critical triangles:

Claim 1. *If x, y, z is a critical triplet of S, then (a) the triangle $\Delta^*(x', y', z')$ is non-degenerated and (b) $x' \in \partial B_\delta(y) \cap \partial B_\delta(z), y' \in \partial B_\delta(z) \cap \partial B_\delta(x)$, and $z' \in \partial B_\delta(x) \cap \partial B_\delta(y)$.*

Proof. The assertion (a) follows from the convexity of balls: if $\Delta^*(x', y', z')$ is degenerated and say $y' \in [x', z']$, since $x', z' \in B_\delta(y)$, from the convexity of $B_\delta(y)$ we conclude that $y' \in B_\delta(y)$, contrary to the assumption that x, y, z is critical.

To prove (b), suppose by way of contradiction that $y' \notin \partial B_\delta(x)$, i.e., $d(x, y') < \delta$. Then there exists an $\epsilon > 0$ such that $B^\circ_\epsilon(y') \subset B_\delta(x)$. On the other hand, the intersection $B^\circ_\epsilon(y') \cap \Delta^*(x', y', z')$ is different from y'. Since $y', x' \in B_\delta(z)$, the convexity of $B_\delta(z)$ implies that $[y', x'] \subset B_\delta(z)$. Therefore, we can find a point $y'' \in [y', x'] \cap B^\circ_\epsilon(y')$ different from y'. Then $y'' \in \Delta^*(x', y', z') \subseteq \Delta^*(x, y, z)$ and y'' still belongs to the intersection $B_\delta(x) \cap B_\delta(z)$. Since $\Delta^*(x', y', z')$ is non-degenerated, by Lemma 11, we obtain $\pi(x', y'', z') < \pi(x', y', z')$, contrary to the choice of the points x', y', z'. This finishes the proof of Claim 1. □

Now, among all triplets of S select a triplet x, y, z for which the perimeter of the critical triangle $\Delta(x', y', z')$ is as large as possible. Notice that such a triplet necessarily exists since the perimeter function $\pi : S \times S \times S \to \mathbb{R}^+$ is continuous because S is convex and attain a maximum because S is compact. Clearly, x, y, z is a critical triplet of S.

Claim 2. $\Delta^*(x', y', z') \subseteq \Delta^*(x^*, y^*, z^*)$. *In particular, $d(x', y'), d(y', z'), d(z', x') \leq \delta$.*

Proof. Since $\Delta^*(x^*, y^*, z^*)$ is convex, it suffices to show that $x', y', z' \in \Delta^*(x^*, y^*, z^*)$. By their definition, the points x', y', z' belong to $\Delta^*(x, y, z)$. The triangle $\Delta^*(x, y, z)$ is the union of four triangles $\Delta^*(x, y^*, z^*), \Delta^*(x^*, y, z^*), \Delta^*(x^*, y^*, z)$, and $\Delta^*(x^*, y^*, z^*)$. Suppose by way of contradiction that one of the points x', y', z' is located in $\Delta^*(x, y^*, z^*) \setminus [y^*, z^*]$. Since $d(x, y^*), d(x, z^*) \leq \delta$, by the convexity of $B_\delta(x)$, $d(x, v) \leq \delta$ for any point $v \in [y^*, z^*]$. Now, if a point w belongs to $\Delta^*(x, y^*, z^*) \setminus [y^*, z^*]$, then extending the geodesic $[x, w]$ through w we will find a point $w' \in [y^*, z^*]$ such that $w \in [x, w']$. Since $d(x, w') \leq \delta$, we conclude that $d(x, w) < \delta$. Consequently, neither of the points x', y', z' can belong to $\Delta^*(x, y^*, z^*) \setminus [y^*, z^*]$ (because each of them belongs to two spheres and does not belong to the third ball). Analogously, one can prove that x', y', z' do not belong to $\Delta^*(x^*, y, z^*) \setminus [x^*, z^*]$ and to $\Delta^*(x^*, y^*, z) \setminus [x^*, y^*]$. Consequently, $x', y', z' \in \Delta^*(x^*, y^*, z^*)$. The second assertion follows from Lemma 12. This establishes Claim 2. □

We continue with a monotonicity property of the shadow $C(x, [y', z'])$. Let $s(y') \in [y, z] \cap [x, y')$ and $s(z') \in [y, z] \cap [x, z')$, where $[x, y')$ and $[x, z')$ are two rays with origin x passing through y' and z', respectively. We will call $s(y')$ and $s(z')$ the *shadows* of y' and z' in $[y, z]$ (or in any line (y, z) extending $[y, z]$). Analogously, one can define the shadow $s(p)$ in $[y, z]$ of any point $p \in [y', z']$ or of any point $p \in \Delta^*(x, y, z)$.

Claim 3. *For any choice of the shadows $s(y')$ and $s(z')$ of y' and z' in $[y, z]$, the points $y, s(z'), s(y'), z$ occur in this order on $[y, z]$.*

Proof. Suppose by way of contradiction that $y, s(y'), s(z'), z$ occur in this order on $[y, z]$. Then $y' \in [x, s(y')] \subset \Delta^*(x, y, s(z'))$. If $y' \in \Delta^*(x, y, z')$, then by Lemma 11 (perimeters of triangles with basis $[x, y]$), we have

$$2\delta < d(x, y') + d(y', y) \leq d(x, z') + d(z', y) = 2\delta,$$

a contradiction. On the other hand, if $y' \in \Delta^*(z', y, s(z'))$, then $[y', z]$ intersects $[x, s(z')]$ and $[z', s(z')]$. Consequently, $z' \in \Delta^*(x, y', z)$ and this case is symmetric to the first case. Since $\Delta^*(x, y, z')$ and $\Delta^*(z', y, s(z'))$ cover $\Delta^*(x, y, s(z'))$, this finishes the proof of Claim 3. □

Claim 4. *If $p, q \in \Delta^*(x, y, z), v \in C(x, [p, q])$, and $v' \in [y, z] \cap (x, v)$, where (x, v) is a line passing via x and v and separating p and q, then there exist shadows $s(p)$ and $s(q)$ of p and q in $[y, z]$ such that $v' \in [s(p), s(q)]$.*

Proof. Pick any shadows $s(p)$ and $s(q)$ of p and q in $[y, z]$. Suppose without loss of generality that the points $y, s(p), s(q), z$ occur in this order on $[y, z]$. Assume that $v' \notin [s(p), s(q)]$, otherwise we are done. Suppose without loss of generality that $v' \in [y, s(p)]$. Since $x, s(p)$, and $s(q)$ all belong to a common closed halfplane defined by $(x, v') = (x, v)$, the whole triangle $\Delta^*(x, s(p), s(q))$ also belong to this halfplane. Since $p, q \in \Delta^*(x, s(p), s(q))$ and the line (x, v) separates p and q, we conclude that $p \in (x, v)$. This implies that $p \in [x, v']$ and consequently, v' is a shadow of p in $[y, z]$. Thus selecting v' as a shadow $s(p)$ of p we are done. □

Claim 5. *The seven triangles*

$$\Delta^*(x, y, z'), \Delta^*(x, y', z), \Delta^*(x', y, z), \Delta^*(x, y', z'), \Delta^*(x', y, z'), \Delta^*(x', y', z), \Delta^*(x', y', z')$$

partition the triangle $\Delta^*(x, y, z)$.

Proof. First we show that $\Delta^*(y,z,x') = \Delta^*(y,z,s_y(x')) \cap \Delta^*(y,z,s_z(x'))$, where $s_y(x')$ and $s_z(x')$ are shadows of x' in $[x,z]$ with respect to y and in $[x,y]$ with respect to z. Indeed, since $x' \in [y,s_y(x')] \cap [z,s_z(x')]$, by convexity of triangles we have $\Delta^*(y,z,x') \subseteq \Delta^*(y,z,s_y(x')) \cap \Delta^*(y,z,s_z(x'))$. To prove the converse inclusion, let $w \in \Delta^*(y,z,s_y(x')) \cap \Delta^*(y,z,s_z(x'))$ and suppose that $w \notin \Delta^*(y,z,x')$. Then $w \in \Delta^*(y,z,s_z(x')) \setminus \Delta^*(y,z,x') = \Delta^*(y,x',s_z(x')) \setminus [y,x']$. Since any shadow of w in $[x,z]$ with respect to y belongs to $[x,s_y(x')] \setminus \{s_y(x')\}$, this contradicts $w \in \Delta^*(y,z,s_y(x'))$. In the same way, we can prove analogous statements for $\Delta^*(x,y,z')$ and $\Delta^*(x,z,y')$. From this and Claim 3 we deduce that the triangles $\Delta^*(y,z,x'), \Delta^*(x,y,z')$, and $\Delta^*(x,z,y')$ pairwise intersect only in the common vertices x,y,z.

Let P be the closure of $\Delta^*(x,y,z) \setminus (\Delta^*(y,z,x') \cup \Delta^*(x,y,z') \cup \Delta^*(x,z,y'))$. Then P is a hexagon with vertices x,y',z,x',y,z' and sides $[x,y'],[y',z],[z,x'],[x',y],[y,z']$, and $[z',x]$. We assert that $[x',y'],[x',z']$, and $[y',z']$ are diagonals of P (i.e., belong to P). If $[x',y']$ is not included in P, then P contains a vertex in $\Delta^*(z,x',y')$ different from z,x',y'. Clearly, this vertex can only be z'. But $\Delta^*(z,x',y') \subseteq B_\delta(z)$ and $d(z,z') > \delta$, a contradiction. The three diagonals do not cross each other because they pairwise have a common extremity. Hence $[x',y'],[x',z'],[y',z']$ triangulate P, concluding the proof of the claim. □

The result of the proposition follows from the following assertion.

Claim 6. $S \subseteq B_\delta(x') \cup B_\delta(y') \cup B_\delta(z')$.

Proof. Pick any point $v \in S$. We distinguish four cases, depending of the location of point v.

Case 1: $v \in \Delta^*(x,y,z)$.

Then v is located in one of the seven triangles defined in Claim 5. First suppose that $v \in \Delta^*(x',y',z')$. Since by Claim 2 each side of $\Delta(x',y',z')$ is of length at most δ, by convexity of balls, $\Delta^*(x',y',z')$ belong to each of the balls $B_\delta(x'), B_\delta(y')$, and $B_\delta(z')$, whence $d(x',v), d(y',v), d(z',v) \leq \delta$.

Now suppose that $v \in \Delta^*(x,y',z') \cup \Delta^*(x',y,z') \cup \Delta^*(x',y',z)$, say $v \in \Delta^*(x,y',z')$. Analogously to the previous case, since the sides of the triangle $\Delta^*(x,y',z')$ are at most δ, we conclude that $d(y',v), d(z',v) \leq \delta$. Finally, suppose that $v \in \Delta^*(x,y,z') \cup \Delta^*(x',y,z) \cup \Delta^*(x,y',z)$, say $v \in \Delta^*(x,y,z')$. Then $x,y \in B_\delta(z')$, whence $v \in \Delta^*(x,y,z') \subseteq B_\delta(z')$, yielding $d(z',v) \leq \delta$. This concludes the proof of Case 1.

Further, we will assume that $v \notin \Delta^*(x,y,z)$.

Case 2: $v \in C(x,[y',z']) \cup C(y,[x',z']) \cup C(z,[x',y'])$.

Suppose without loss of generality that v belongs to the shadow $C(x,[y',z'])$. If $x' \in [x,v]$, then $d(x',v) = d(x,v) - d(x,x') \leq \delta$ and we are done, hence we assume from now that $x' \notin [x,v]$. We have $[x,v] \cap [y',z'] \neq \varnothing$. Then necessarily $[x,v]$ intersects one of the sides $[z',x']$ and $[y',x']$ of $\Delta^*(x',y',z')$, say $[z',x']$. But then $[x,v]$ intersects $\partial B_\delta(x) \cap \Delta^*(x',y',z')$ in a point v' and $\partial B_\delta(y) \cap \Delta^*(x',y',z')$ in a point v'', where $v' \in [x,v'']$. Since $d(v,x) \leq 2\delta$ and $d(x,v') = \delta$, we conclude that $d(v,v'') \leq d(v,v') \leq 2R - R = R$.

Next, we assert that $[y,x'] \cap [v,v''] \neq \varnothing$. Let $s(x')$ be a shadow of x' on $[y,z]$; we may assume that $[x,v] \cap [y,s(x')] \neq \varnothing$. Then considering $\Delta^*(y,s(x'),x')$, the geodesic $[x,v]$ intersects another its side, either $[y, x']$ or $[x', s(x')]$. In the latter case, it follows that $x' \in [x, v]$ and we excluded that case. Hence we can assume the former case. Then as v'' is not in the interior of $\Delta^*(y, x', z)$ by Claim 5. $[v'', v] \cap [y, x'] \neq \varnothing$, as asserted.

Hence, we can suppose that $[y, x'] \cap [v, v''] \neq \varnothing$. By Lemma 13, $d(y, v'') + d(v, x') \leq d(y, x') + d(v, v'')$. Since $d(y, v'') = d(y, x') = \delta$ and $d(v, v'') \leq \delta$, we obtain that $d(v, x') \leq \delta$, concluding the proof of Case 2.

Case 3: $v \in C(x, \Delta^*(x', y', z')) \cup C(y, \Delta^*(x', y', z')) \cup C(z, \Delta^*(x', y', z'))$.

Suppose without loss of generality that $v \in C(x, \Delta^*(x', y', z'))$. In view of Case 2, we can assume that $v \notin C(x, [y', z'])$. Since any point of $C(x, \Delta^*(x', y', z'))$ belongs to two of the three shadows $C(x, [x', y']), C(x, [x', z'])$, and $C(x, [y', z'])$, necessarily $v \in C(x, [x', y']) \cap C(x, [x', z'])$. By definition of $C(x, [x', z'])$, there is a line (x, v) passing via x and v and separating x' from z'. Let $v' \in [x', z'] \cap (x, v)$. Let $s(v')$ be a shadow of v' in $[y, z]$ such that $s(v') \in [x, v] \cap [y, z]$ (it exists because $v' \in [y, v]$). Notice that $s(v') \notin C(x, [y', z'])$. Indeed, otherwise there exists a line $(x, s(v'))$ extending $[x, s(v')]$ and separating the points y' and z'. But then $[x, s(v')]$ separates y' and z' in $\Delta^*(x, y, z)$. Therefore any line extending $[x, s(v')]$, in particular the line (x, v), also separates the points y' and z'. This contradicts the assumption $v \notin C(x, [y', z'])$. Hence $s(v') \notin C(x, [y', z'])$.

Consider the shadows $s(x'), s(y')$, and $s(z')$ of x', y', and z' in $[y, z]$ such that $s(v') \in [s(x'), s(z')]$ (such shadows $s(x')$ and $s(z')$ exist by Claim 4). Since $C(x, [y', z'])$ is convex (Lemma 16) and $s(v') \notin C(x, [y', z'])$, we conclude that $s(v')$ does not belong to $[s(z'), s(y')]$. By Claim 3, either $s(v')$ belongs to $[y, s(z')]$ or $s(v')$ belongs to $[s(y'), z]$, say the first. Consequently, further we will assume that $s(v') \in [y, s(z')]$ and $s(v') \neq s(z')$. We have:

- (i) $d(y, v') \leq \delta$, because $v' \in [x', z']$, $d(y, x') = d(y, z') = \delta$ and $B_\delta(y)$ is convex (Lemma 5),
- (ii) $d(v, v') = d(v, x) - d(v', x) \leq 2\delta - \delta = \delta$, because $v' \in [v, x]$ and $d(v', x) > \delta$ by minimality of $\pi(x', y', z') \geq \pi(x', y', v')$.

Assume now that $x' \in \Delta^*(y, v, v')$. Then applying Lemma 11 to the triangles $\Delta^*(y, v, x')$ and $\Delta^*(y, v, z')$ having $[y, v]$ as a side, we obtain $d(y, x') + d(x', v) \leq d(y, v') + d(v', v)$. Since $d(y, x') = \delta$ and $d(y, v'), d(v', v) \leq \delta$, we derive that $d(x', v) \leq \delta$.

It remains to prove that $x' \in \Delta^*(y, v, v')$. We prove this in two steps. First, we show that $x' \in \Delta^*(x, y, v)$. Since $s(v') \in [y, s(z')]$ and $s(v') \neq s(z')$, the point v' belongs to $\Delta^*(y, x, s(z')) \setminus [x, s(z')]$. Since $v' \in [x', z']$, we conclude that x' also belongs to $\Delta^*(y, x, s(z')) \setminus [x, s(z')]$. Moreover, since $s(v') \in [s(x'), s(z')]$, the point $s(x')$ is located between y and $s(v')$. Since $x' \in [s(x'), x]$, x' belongs to the triangle $\Delta^*(x, y, s(v'))$ and therefore to the triangle $\Delta^*(x, y, v)$.

Second, we prove by way of contradiction that $x' \notin \Delta^*(x, y, v')$. Otherwise, if $x' \in \Delta^*(x, y, v')$, let z'' be a point in the intersection of $[x, y]$ and a geodesic line extending $[x', v'] \subseteq [x', z']$. Then $v' \in [z', z''] \subset \Delta^*(x, y, z')$. Applying Lemma 11 to the triangles $\Delta^*(x, y, z')$ and $\Delta^*(x, y, x')$ having $[x, y]$ as a side, we get $2\delta < d(y, x') + d(x', x) \leq d(y, z') + d(z', x) = 2\delta$, a contradiction. This shows that indeed $x' \in \Delta^*(y, v, v')$ and concludes the proof of Case 3.

Case 4: $v \notin C(x,\Delta^*(x',y',z')) \cup C(y,\Delta^*(x',y',z')) \cup C(z,\Delta^*(x',y',z'))$.

Suppose without loss of generality that v is separated from $\Delta^*(x',y',z')$ by a line (y,z) extending $[y,z]$ (such a line exists by Lemma 17). Suppose also by way of contradiction that $v \notin B_\delta(x') \cup B_\delta(y') \cup B_\delta(z')$. Since the shadow $C(x,\Delta^*(x',y',z'))$ is convex by Lemma 16, the intersection of $C(x,\Delta^*(x',y',z'))$ with (y,z) (and with $[y,z]$) is a geodesic segment $[p,q]$. Let $v' \in [x,v] \cap (y,z)$. We assert that $v' \notin [p,q]$. Indeed, if $v' \in [p,q]$, then $v' \in C(x,\Delta^*(x',y',z'))$, thus the intersection $[x,v'] \cap \Delta^*(x',y',z')$ is nonempty. Since $[x,v'] \subseteq [x,v]$, we conclude that $[x,v] \cap \Delta^*(x',y',z') \neq \varnothing$, contrary to our assumption that $v \notin C(x,\Delta^*(x',y',z'))$. Consequently, $v' \notin [p,q]$. Then one can easily see that either $[p,q] \subseteq [y,v']$ or $[p,q] \subseteq [v',z]$ holds, say the first. In this case, since $s(x'), s(y'), s(z') \in [p,q]$, and $x' \in [x,s(x')], y' \in [x,s(y')], z' \in [x,s(z')]$, we deduce that $x',y',z' \in \Delta^*(x,y,v)$. This shows that either $\Delta^*(x',y',z') \subseteq \Delta^*(x,y,v)$ or $\Delta^*(x',y',z') \subseteq \Delta^*(x,z,v)$ holds, say the first.

Let $\Delta^*(x'',y'',v'')$ be the critical triangle of the triplet x,y,v. We assert that $x',y',z' \in \Delta^*(x'',y'',v'')$. For this we will first prove that

$$\Delta^*(x,y,v) \setminus (B^\circ_\delta(x) \cup B^\circ_\delta(y) \cup B^\circ_\delta(v)) \subseteq \Delta^*(x'',y'',v'') \setminus [x'',y''].$$

Indeed, since $d(y,x'') = d(y,v'') = \delta$ and the balls are convex, $\Delta^*(y,v'',x'') \subseteq B_\delta(y)$. Moreover, $\Delta^*(y,v'',x'') \setminus [v'',z''] \subseteq B^\circ_\delta(y)$. Indeed, any point $p \in \Delta^*(y,v'',x'') \setminus [v'',z'']$ belongs to a geodesic segment $[y,q]$ with $q \in [v'',z'']$. Since $q \in B^\circ_\delta(y)$ and $p \neq q$, necessarily $d(y,p) < \delta$. Analogously, we obtain that $\Delta^*(y'',x'',v) \setminus [y'',x''] \subseteq B^\circ_\delta(v)$ and $\Delta^*(x,v'',y'') \setminus [v'',y''] \subseteq B^\circ_\delta(x)$. On the other hand, each of the triangles $\Delta^*(x,y,v''), \Delta^*(x,y'',v)$, and $\Delta^*(x'',y,v)$ is covered by two of the three open balls $B^\circ_\delta(x), B^\circ_\delta(y)$, and $B^\circ_\delta(v)$. For example, $\Delta^*(x,y,v'')$ is covered by $B^\circ_\delta(x)$ and $B^\circ_\delta(y)$. Indeed, by monotonicity of perimeters (Lemma 11), for any point $p \in \Delta^*(x,y,v'')$, we have $\min\{d(p,x), d(p,y)\} \leq \delta$. Moreover, by the same result, if $d(x,p) = \delta$, then $d(y,p) < \delta$. This establishes that $\Delta^*(x,y,v'') \subseteq B^\circ_\delta(x) \cup B^\circ_\delta(y)$. Now, the required inclusion follows from Claim 5.

Since z' has distance δ to x and y and z' has distance $> \delta$ to z and v, from previous inclusion we obtain $z' \in \Delta^*(x'',y'',v'')$. Analogously, since x' has distance δ to y and z and x' has distance $> \delta$ to x and v, we conclude that $x' \in \Delta^*(x'',y'',v'')$ (the proof for y' is analogous). Hence $x',y',z' \in \Delta^*(x'',y'',z'')$. From Lemma 11 we conclude that $\pi(x',y',z') < \pi(x'',y'',v'')$, contrary to the choice of the triplet x,y,z as a triplet having a critical triangle $\Delta^*(x',y',z')$ of maximal perimeter. This concludes the proof of Claim 6 and of Proposition 2. □

3.4. **Proof of Proposition 3.** Let S be a compact subset of a Busemann surface $(\mathcal{S},d)$. Let u,v be a diametral pair of S, i.e., $u,v \in S$ and $d(u,v) = \mathrm{diam}(S)$. Let $\ell := (u,v)$ be a line extending $[u,v]$ and let S' and S'' be the intersections of S with the closed halfplanes Π'_ℓ and Π''_ℓ defined by ℓ. We will show how to cover each of the sets $S'_0 := S' \cap B_{2\delta}(v)$ and $S''_0 := S'' \cap B_{2\delta}(v)$ with a fixed number of balls of radius δ. We will establish this for S'_0, the same method works for S''_0; at the end we will optimize over the two solutions since some balls from different solutions have the same centers and thus coincide.

If $\mathrm{diam}(S) \leq 2\delta$, we simply apply Proposition 2. Therefore, further we will assume that $\mathrm{diam}(S) > 2\delta$. By Lemma 15, u,v is also a diametral pair of $\mathrm{conv}(S)$ and of $\mathrm{conv}(S')$. Let x

be a point of $[u,v]$ at distance 2δ from u. Let w be a point of $\mathrm{conv}(S') \cap \partial B_{2\delta}(v)$ maximizing the distance to u, i.e., maximizing the perimeter $\pi(u,v,w)$. Such a point w exists because the set $\mathrm{conv}(S') \cap \partial B_{2\delta}(v)$ is compact and nonempty (the point x belongs to this intersection).

Let x' be a point of $[u,w]$ at distance $\left(1 - \frac{2\delta}{d(u,v)}\right) d(u,w)$ from w. Notice that since $d(u,w) \leq d(u,v)$, we have $d(x',w) \leq 2\delta$. Notice also that if we set $t := 1 - \frac{2\delta}{d(u,v)}$, then $0 < t < 1$ and x is the point of $[u,v]$ such that $d(u,x) = t \cdot d(u,v)$ and x' is the point of $[u,w]$ such that $d(u,x') = t \cdot d(u,w)$. By Lemma 3(ii), $d(x,x') \leq t \cdot d(v,w) < d(v,w) \leq 2\delta$. On the other hand, $d(u,x) - d(u,x') = t \cdot (d(u,v) - d(u,w)) \geq 0$. Since $d(x,v) = 2\delta$, we conclude that $d(v,x') \geq 2\delta$ and equality $d(v,x') = 2\delta$ holds if and only if $x = x'$ (because in case of equality, x and x' belong to the geodesic $[u,v]$ and thus they must coincide). Let A be the quadrilateral of $\Delta^*(u,v,w)$ bounded by the four geodesics $[x,x'], [x',w], [w,v]$, and $[v,x]$.

Claim 1. *$\Delta^*(u,v,w) \cap S'_0 = A \cap S'_0$.*

Proof. Indeed, suppose by way of contradiction that there exists a point $z \in \Delta^*(u,v,w) \cap S'_0$ not belonging to A. Let z' be a point obtained as the intersection of $[x,x']$ with the extension of the geodesic $[u,z]$ through z. Then $z \in [u,z']$ and $z' \neq z$, yielding $d(u,z) < d(u,z')$. Since $d(u,z') \leq \max\{d(u,x), d(u,x')\} = d(u,x)$ by the convexity of balls, we deduce that $d(u,z) < d(u,x)$. Since $d(v,z) \leq 2\delta$ and $d(v,x) = 2\delta$, we conclude that $d(u,z) + d(z,v) < d(u,x) + d(x,v)$, contrary to the choice of x from $[u,v]$. This finishes the proof of Claim 1. □

Let B be the region of the halfplane Π' consisting of all points z such that $[u,z] \cap [v,w] \neq \varnothing$. Finally, let C be the region of Π' consisting of all points z such that $[z,v] \cap [u,w] \neq \varnothing$. Notice that $B \cup C$ consists of precisely those points z of Π' such that $\Delta^*(u,v,w)$ and $\Delta^*(u,v,z)$ are not comparable.

Claim 2. *$S'_0 \subseteq A \cup B \cup C$.*

Proof. Suppose by way of contradiction that S'_0 contains a point z such that $\Delta^*(u,v,w)$ is properly included in $\Delta^*(u,v,z)$. If $d(v,z) = 2\delta$, then $\pi(u,v,w) < \pi(u,v,z)$ by Lemma 11, and we will obtain a contradiction with the choice of w. Thus $d(v,z) < 2\delta$. Since $u \notin B_{2\delta}(v)$, the geodesic $[u,z]$ intersects $\partial B_{2\delta}(v)$ in a point w'. Let w'' be a common point of $[u,z]$ and a geodesic extension (v,w). Then $w \in [w'',v]$. Since $d(v,w) = 2\delta$, we have $d(v,w'') > 2\delta$. Since w' and w'' are located on $[u,z]$, $d(v,z) \leq 2\delta$, and $d(v,w') = 2\delta$, the convexity of $B_{2\delta}(v)$ implies that w' is located on $[u,z]$ between w'' and z. This means that $\Delta^*(u,v,w)$ is properly contained in $\Delta^*(u,v,w')$. By Lemma 11, $\pi(u,v,w') > d(u,v,w)$. Now, since $w' \in [z,u]$, $d(v,w') = 2\delta$, and $z \in S'_0$, we conclude that $w' \in \mathrm{conv}(S') \cap \partial B_{2\delta}(v)$, contradicting the choice of w. This finishes the proof of Claim 2. □

Now, we will analyze how to cover the points of S'_0 in each of the regions A, B, C.

Claim 3. *$\mathrm{diam}(B \cap S'_0) \leq 2\delta$.*

Proof. Pick any two points $y, y' \in B \cap S'_0 \leq 2\delta$. If the triangles $\Delta^*(u,v,y)$ and $\Delta^*(u,v,y')$ are incomparable, i.e., $y \notin \Delta^*(u,v,y')$ and $y' \notin \Delta^*(u,v,y)$, then $[y,v] \cap [y',u] \neq \varnothing$ or

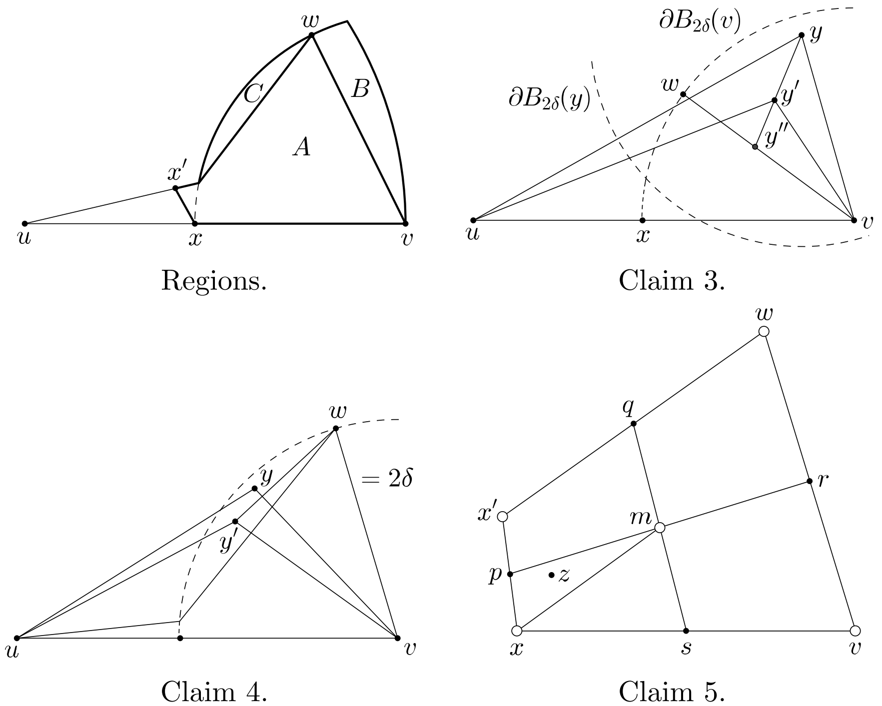

FIGURE 2. Illustration of the proof of Proposition 3.

$[y', v] \cap [y, u] \neq \varnothing$, say the first (this dichotomy follows from the fact that $\mathcal{S}$ is homeomorphic to $\mathbb{R}^2$). By Lemma 13, $d(y, y') + d(u, v) \leq d(y, v) + d(u, y')$. Since $d(y, v) \leq 2\delta$ and $d(u, y') \leq d(u, v)$ (by the choice of v), we conclude that $d(y, y') \leq d(y, v) \leq 2\delta$.

Now, suppose that $y' \in \Delta^*(u, v, y)$. Since $[y, u]$ intersects $[v, w]$ and $d(u, y) \leq d(u, v)$ by the choice of v, by Lemma 13 we have $d(y, w) \leq d(v, w) = 2\delta$. Also $d(v, y) \leq 2\delta$ because $y, y' \in S'_0$. Since $v, w \in B_{2\delta}(y)$, by the convexity of the ball $B_{2\delta}(y)$ we conclude that $y' \in B_{2\delta}(y)$. Hence $d(y, y') \leq 2\delta$. Consequently, $\operatorname{diam}(B \cap S'_0) \leq 2\delta$. □

Claim 4. $\operatorname{diam}(C \cap S'_0) \leq 2\delta$.

Proof. Pick any two points $y, y' \in C \cap S'_0$. Again, if the triangles $\Delta^*(u, v, y)$ and $\Delta^*(u, v, y')$ are incomparable, then we proceed as in the proof of Claim 1. Now suppose that $y' \in \Delta^*(u, v, y)$. Since $[y', v]$ intersects $[u, w]$ and $d(v, y') \leq 2\delta$, $d(u, w) \leq d(u, v)$, from Lemma 13 we deduce that $d(y', w) \leq 2\delta$. Since $y' \in \Delta^*(u, v, y) \setminus \Delta^*(u, v, w)$, we conclude that $[y, v] \cap [y', w] \neq \varnothing$. Again, by Lemma 13 $d(y, y') + d(v, w) \leq d(y, v) + d(y', w)$. Since $d(v, w) = 2\delta$, $d(y, v), d(y', w) \leq 2\delta$, we immediately conclude that $d(y, y') \leq 2\delta$. □

Claim 5. *The set A and consequently the set $S'_0 \cap A$ can be covered by 4 balls of radius δ.*

Proof. Recall that A is a convex quadrilateral having all four sides $[x, x']$, $[x', w]$, $[w, v]$, and $[v, x]$ of size at most 2δ. Let p, q, r, and s be the midpoints of $[x, x']$, $[x', w]$, $[w, v]$, and $[v, x]$, respectively. By Lemma 3(i), $d(p, r) \leq 2\delta$ and $d(q, s) \leq 2\delta$. Let m be the midpoint of $[q, s]$.

Again, by Lemma 3(i), $d(p,m) \le \delta$ and $d(m,r) \le \delta$. Since $d(m,q), d(m,s) \le \delta$, the geodesics $[m,p], [m,q], [m,r]$, and $[m,s]$ partition A into four convex quadrilaterals with all sides at most δ.

We assert that A is covered by the four δ-balls centered at the points p, q, r and s. Indeed, pick any point z of A. Without loss of generality, we show that the quadrilateral with vertices x, p, m, and s is covered by $B_\delta(p)$ and $B_\delta(q)$. The geodesic $[x,m]$ splits this quadrilateral into two triangles $\Delta^*(x,p,m)$ and $\Delta^*(x,s,m)$. By convexity of balls, we have $Delta^*(x,p,m) \subseteq B_\delta(p)$ and $\Delta^*(x,m,s) \subseteq B_\delta(s)$. □

Summarizing, we conclude that S_0' can be covered by $3+3+4 = 10$ balls of radius δ. Analogously, the set S_0'' can be covered by 10 balls of radius δ. However, notice that the ball $B_\delta(s)$ is counted in both coverings, thus $S \cap B_{2\delta}(v)$ can be covered by 19balls of radius δ. This finishes the proof of Proposition 3.

3.5. **Proof of Corollary 1.** Let $\mathcal{P}$ be a simple polygon endowed with the geodesic metric. Then we can triangulate $\mathcal{P}$ and consider $\mathcal{P}$ as a 2-dimensional cell complex, whose 2-cells are the triangles of the triangulation. In [13] we showed how to extend $\mathcal{P}$ to a Busemann surface $(\mathcal{S}, d)$. Notice that in fact by this construction $\mathcal{P}$ is embedded as a convex subset of $(\mathcal{S}, d)$. Notice also that the resulting $(\mathcal{S}, d)$ is not only Busemann but also a CAT(0) space. By Theorem 1, $\rho(\mathcal{P}) \le 19\nu(\mathcal{P})$. Let $\mathcal{C} = \{B_\delta(x_1), \ldots, B_\delta(x_k)\}$ be any covering of $\mathcal{P}$ with closed δ-balls of $(\mathcal{S}, d)$. We will show how to transform $\mathcal{C}$ into a covering $\mathcal{C}' = \{B_\delta(x_1'), \ldots, B_\delta(x_k')\}$ of the same size in which all balls have their centers in $\mathcal{P}$. Pick any center x_i and consider its metric projection (nearest point) in $\mathcal{P}$. Since $(\mathcal{S}, d)$ is CAT(0) and $\mathcal{P}$ is a compact convex subset of $\mathcal{S}$, by [8, Proposition 2.4(1)], this projection is unique; denote it by x_i'. By [8, Proposition 2.4(4)], for any point $y \in \mathcal{P}$, $d(x_i', y) \le d(x_i, y)$. Hence $B_\delta(x_i) \cap \mathcal{P} \subseteq B_\delta(x_i')$ and we can replace x_i by its projection x_i'.

4. Open questions

We conclude the paper with three open questions.

Question 4.1. Describe a polynomial time algorithm (in the number of sides and the size of the packing) that, given a simple polygon $\mathcal{P}$ with n sides, constructs a covering and a packing of $\mathcal{P}$ satisfying the conditions of Corollary 1. Equivalently, find a polynomial in n algorithm (and maybe in the description of S) to implement each step of the algorithm resulting from Propositions 1-3: finding a covering of a closed subset S of $\mathcal{P}$ of diameter $\le 2\delta$ with at most 3 balls (Proposition 2) and the construction of the regions A, B, and C in the proof of Proposition 3.

Question 4.2. Is it true that there exists a universal constant c such that $\rho(S) \le c\nu(S)$ for any compact (finite) subset of points of an arbitrary polygon (with holes) endowed with the geodesic metric? Does such a constant c exist if $\mathrm{diam}(S) \le 2\delta$, i.e., do polygons with holes satisfy the weak-doubling property? The same questions can be raised for polygons with holes on Busemann surfaces.

Question 4.3. Is it true that the results of this note can be extended to all 2-dimensional Busemann spaces and, more generally, to all n-dimensional Busemann spaces (in the latter case, the constant c will depend of n)? The case of CAT(0) cube complexes (and, in particular, of CAT(0) square complexes) is already interesting and nontrivial.

References

[1] P.K. Agarwal and N.H. Mustafa, Independent set of intersection graphs of convex objects in 2D, Computational Geometry, 34 (2006), 83–95.

[2] N. Alon, Piercing d-intervals, Discrete & Computational Geometry, 19 (1998), 333–334.

[3] N. Alon, Covering a hypergraph of subgraphs, Discrete Mathematics, 257 (2002), 249–254.

[4] I. Bárány, J. Edmonds, and L. A. Wolsey, Packing and covering a tree by subtrees, Combinatorica, 6 (1986), 221–233.

[5] C.Berge, Hypergraphs, North-Holland, Amsterdam, 1989.

[6] G. Borradaile and E.W. Chambers, Covering nearly surface-embedded graphs with a fixed number of balls, Discrete & Computational Geometry, 51 (2014), 979–996.

[7] K. Böröczky, Finite Packing and Covering, Cambridge University Press, Cambridge, 2004.

[8] M.R. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Grundlehren der Mathematischen Wissenschaften vol. 319, Springer-Verlag, Berlin, 1999.

[9] H. Brönnimann and M. Goodrich, Almost optimal set covers in finite VC-dimension, Discrete & Computational Geometry, 14(1995), 463—479.

[10] N. Bousquet, Hitting Sets: VC-dimension and Multicut, PhD Thesis, Université de Montpellier II, 2013.

[11] N. Bousquet and S. Thomassé, VC-dimension and Erdös-Pósa property, Discrete Mathematics, 338 (2015), 2302–2317.

[12] P. Cameron, Problems from CGCS Luminy, May 2007, European Journal of Combinatorics 31 (2010), 644—648.

[13] J. Chalopin, V. Chepoi, and G. Naves, Isometric embedding of Busemann surfaces into L_1, Discrete & Computational Geometry, 53 (2015), 16–37.

[14] T. Chan and S. Har-Peled, Approximation algorithms for maximum independent set of pseudo-disks, Discrete & Computational Geometry, 48 (2012), 373–392.

[15] V. Chepoi and B. Estellon, Packing and covering delta-hyperbolic spaces by balls, APPROX-RANDOM'07, pp. 59–73, August 20-22, 2007, Princeton, USA.

[16] V. Chepoi, B. Estellon, and Y. Vaxès, On covering planar graphs with a fixed number of balls, Discrete & Computational Geometry, 37 (2007), 237–244.

[17] V. Chepoi and S. Felsner, Approximating hitting sets of axis-parallel rectangles with opposite corners separated by a monotone curve, Computational Geometry, 46 (2013), 1036–1041.

[18] K.L. Clarkson, Nearest neighbor queries in metric spaces, Discrete & Computational Geometry 22 (1999), 63–93.

[19] G. Cornuejols, Combinatorial Optimization: Packing and Covering, Society for Industrial and Applied Mathematics, 2001.

[20] J.R. Correa, L. Feuilloley, P. Pérez-Lantero, and J.A. Soto, Independent and hitting sets of rectangles intersecting a diagonal line: algorithms and complexity, Discrete & Computational Geometry, 53 (2015), 344–365.

[21] R.M. Dudley, Uniform Central Limit Theorems, Cambridge University Press, Cambridge, 1999.

[22] H. Hadwiger and H. Debrunner, Kombinatorische Geometrie in der Ebene, Inst. Math., Genève 1959.

[23] A. Gyárfás and J. Lehel, Covering and coloring problems for relatives of intervals, Discrete Mathematics, 55 (1985), 167–180.

[24] S. Ivanov, On Helly's theorem in geodesic spaces, Electronic Research Announcements in Mathematical Sciences, 21 (2014), 109–112.

[25] A. N. Kolmogorov and V. M. Tihomirov, ϵ-Entropy and ϵ-capacity of sets in function spaces, Uspehi Mat. Nauk, 14 (1959), 3–86, English transl. in Amer. Math. Soc. Transl. 17, 277–364.

[26] S.R. Kulkarni and S.K. Mitter, On Metric entropy, Vapnik-Chervonenkis dimension, and learnability for a class of distributions, AAAI Symposium on Minimal Length Encoding, Stanford, March, 1990. Also, Center for Intelligent Control Systems Report CICS-160, M.I.T., 1989.

[27] G.G. Lorentz, Metric entropy and approximation, Bulletin of American Mathematical Society, 72 (1966), 903–937.

[28] N. Mustafa and S. Ray, Improved Results on Geometric Hitting Set Problems, Discrete & Computational Geometry, 44 (2010), 883–895.

[29] J. Pach and P. Agarwal, Combinatorial Geometry, John Wiley & Sons, New York, NY, 1995.

[30] A. Papadopoulos, Metric Spaces, Convexity and Nonpositive Curvature, IRMA Lectures in Mathematics and Theoretical Physics, vol. 6, European Mathematical Society, 2005.

[31] R. Pollack, M. Sharir, and G. Rote. Computing the geodesic center of a simple polygon, Discrete & Computational Geometry, 4 (1989), 611–626.

[32] A. Schrijver, Combinatorial Optimization, Vol. B, Springer-Verlag, Berlin, 2003.

[33] M. van de Vel, Theory of Convex Structures, Elsevier, Amsterdam, 1993.

[34] V. Vazirani, Approximation Algorithms, Springer, 2001.

[35] I. Vigan, Packing and covering a polygon with geodesic disks, arXiv:1311.6033v1 (2013).

[36] G. Wegner, Über eine kombinatorisch-geometrische Frage von Hadwiger and Debrunner, Israel Journal of mathematics, 3 (1965), 187–198.