

# Congestion in planar graphs with demands on faces

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We show the following theorem:

**Theorem 0.1.** *The congestion in  $(G, H, r, c)$  is at most  $2\lceil \log_2 k \rceil + 2$  when  $G$  is an embedded planar graph, each demand  $h \in H$  lies on a face of  $G$ , and there are at most  $k$  terminals in each face of  $G$ .*

We will use the celebrated theorem of Seymour:

**Theorem 0.2.** *Let  $(G, H, r, c)$  be an instance of the multifold problem such that  $G + H$  has no  $K_5$ -minor. Then the cut condition is equivalent to the existence of a half-integer multifold. If  $G + H$  is Eulerian, the cut condition is equivalent to the existence of an integer multifold.*

Note that by Kuratowski's theorem, planar graphs have no  $K_5$  minor.

## 1 Proof

Let  $G$  be a planar graph. Without loss of generality, we suppose that  $G$  is 2-connected. This means that the boundaries of its faces are circuits. For any instance  $(G, H, r, c)$ , we define  $r(e) = 0$  for every edge  $e \notin E(H)$ .

We say that two demand  $s_1t_1, s_2t_2$  are crossed if they both lie on the same face of  $G$  and  $s_1, s_2, t_1, t_2$  appears in that order around the boundary of the face. We call *uncrossing*  $(G, H)$  by  $s_1t_1, s_2t_2$  and denote  $(G, H, r, c) \oplus (s_1t_1, s_2t_2)$  the instance  $(G, H', r', c)$  where:

- $r'(s_1t_1) = r(s_1t_1) - m$  and  $r'(s_2t_2) = r(s_2t_2) - m$ , where  $m = \min\{r(s_1t_1), r(s_2t_2)\}$ ,
- $r'(s_1s_2) = r(s_1s_2) + m$  and  $r'(t_1t_2) = r(t_1t_2) + m$ ,
- $r'(e) = r(e)$  for every other edge  $e$ ,
- $H' = \{uv : r'(uv) > 0\}$ .

**Lemma 1.1.** *Let  $G$  be an embedded planar graph,  $H$  a demand graph for  $G$ , and  $s_1t_1, s_2t_2$  two demands of  $H$  lying on the same face of  $G$ . If  $(G, H, r, c)$  satisfies the cut condition, so does  $(G, H, r, c) \oplus (s_1t_1, s_2t_2)$ .*

*Proof.* It follows from the fact that the cut condition is satisfied iff it is satisfied for central cuts only. But the intersection of a central cut and the boundary of a face is a path.  $\square$

As a consequence, for any set of disjoint crossed demand edges, the cut condition for  $(G, H)$  implies the cut condition for the uncrossing of  $(G, H)$  by these crossed demand edges.

From now on, we suppose the cut condition is satisfied by  $(G, H, r, c)$ . Let  $F$  be any face of  $G$  that contains some demand edges  $H_F$ . If  $G + H_f$  is planar, then by doubling  $r$  and  $c$  and applying the Eulerian part of Theorem 0.2,  $G + H_f$  has congestion two. Note that actually, for any number of faces  $H_{F_1}, \dots, H_{F_i}$ , if  $G + H_{F_1} + \dots + H_{F_i}$  is planar, its congestion is 2. For convenience, we will only look at one face at a time, but all the arguments can (and must) be applied simultaneously on all the faces. We use this principle to decrease by half the maximum number of terminals on one face of the demand graph. Note that when a face  $F$  has a single or no demand,  $G + H_F$  is obviously planar.

Let  $F$  be a face with a least two demands. For convenience, we only consider the vertices of the boundary of  $F$  that are terminals of the demand lying in  $F$ , call them  $u_1, u_2, \dots, u_m$  (in the order of appearance on the boundary), where  $m = |V(H_F)|$ . Let  $k = \lfloor \frac{m}{2} \rfloor$ . A demand edge is *bilateral* if one of its extremity is in  $R = \{u_1, \dots, u_k\}$  and the other is in  $L = \{u_{k+1}, \dots, u_m\}$ . We want to route all the bilateral demands with a congestion of 2. Then we would add an edge of capacity 0 between  $u_k$  and  $u_m$ , completing the proof. Actually, we will not solve these demands, but we will uncross all of them in such a way that the new demands will have their two extremities both in  $L$  or both in  $R$ .

We define iteratively crossed pairs of bilateral edges of  $H_F$ . Let  $i$  be the minimum index such that there is a bilateral edge  $u_i u_j$  in  $H_F$ . Let  $j'$  be the maximum index such that there is a bilateral edge  $u_{i'} u_{j'}$  in  $H_f$ . Note that  $i$  exists iff  $j'$  exists. Let  $m = \min\{r(u_i u_j), r(u_{i'} u_{j'})\}$ . We distinguish two cases:

- either  $i = i'$  and  $j = j'$ , then we mark  $u_i u_j$  in white,
- or we select the crossed edges  $u_i u_j$  and  $u_{i'} u_{j'}$ , and mark  $u_i u_{j'}$  in white.

In both cases, we decrease the requests on the edges  $u_i v_j$  and  $v_k, u_l$  by  $m$  and remove the demand edges with capacity 0. We repeat this procedure until there is no more edges between  $u_1, \dots, u_m$  and  $v_1, \dots, v_m$ .

Thus, we have a set  $S$  of selected crossed disjoint pairs of demands and a set  $W$  of white edges. By induction, it is easy to see that there are no two crossed white edges. Moreover, by Lemma 1.1, the two following instances satisfies the cut condition:

$$(i) (G, H, r, c) \oplus \bigoplus_{(u_i u_j, u_{i'} u_{j'}) \in S} (u_i u_{j'}, u_{i'} u_j)$$

$$(ii) (G, H, r, c) \oplus \bigoplus_{(u_i u_j, u_{i'} u_{j'}) \in S} (u_i u_{i'}, u_j u_{j'})$$

By (i),  $(G, W)$  also satisfies the cut condition (by simply removing the non-white demand edges). By Theorem 0.2,  $(2G, 2W)$  admits an integer solution. From this solution, we only keep two paths for each unit of capacity of

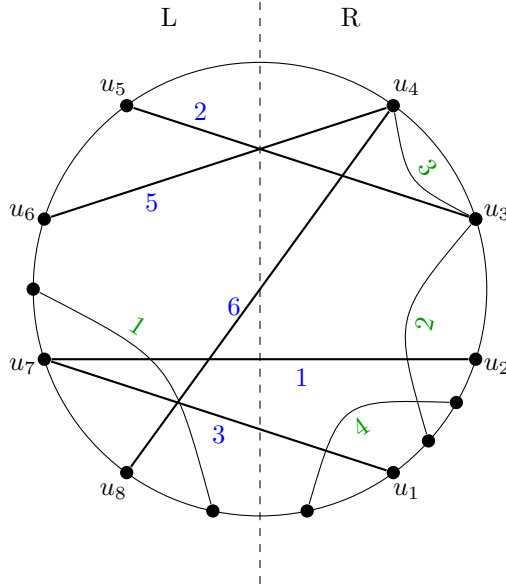


Figure 1: The original face. The capacities of the bilateral demands are in blue, other demands are in green.

the edge  $u_i u_{j'}$ , for each  $(u_i u_j, u_{i'} u_{j'}) \in S$ . For all the edges  $u_i v_j \in W \setminus E(S)$ , we keep as many path as the capacity. This means that now we only have to find paths for each of the demands  $u_i u_{i'}$  and  $u_j u_{j'}$  (and combine them with the two  $(u_i, u_{j'})$ -paths), for each selected pair  $(u_i u_j, u_{i'} u_{j'})$ , plus paths for all the non-bilateral demands. It corresponds to (ii) without the edges in  $W \setminus E(S)$ , thus it satisfies the cut condition, and there is no bilateral demand edge. By adding one supply edge with capacity 0 between  $u_m$  and  $u_k$  (it obviously does not violate the cut condition, nor does it changes the feasibility of the instance), we obtain two new faces with at most half the number of terminals of the original face.

By applying this procedure simultaneously (that is with only one invocation of Theorem 0.2) to every face, the maximal number of terminals in one face is divided by two. Now, by induction, as each step uses  $2G$ , the Theorem 0.1 is proved.

## 2 Lower bound

We now prove that one cannot largely improve our bound on congestion by simply using Seymour's Theorem 0.2 as we did. More precisely, suppose we apply Theorem 0.2  $c$  times to a face  $F$  containing a set  $T$  of  $n$  terminals. Without loss of generality, we prove the bound for the case when  $H_F$  is a matching. For each application, we get a solution to a planar demand graph on  $F$ , with at most

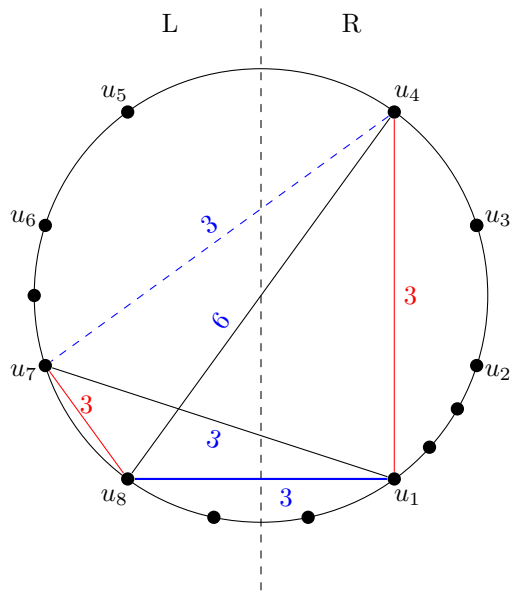


Figure 2: First iteration,  $i = 1$ ,  $j = 7$ ,  $j' = 8$  and  $i' = 4$ . The minimum demand here is 3, we decrease the capacities of these two bilateral demands by 3. The blue continuous edge is marked white. The red edges are the result of the uncrossing.

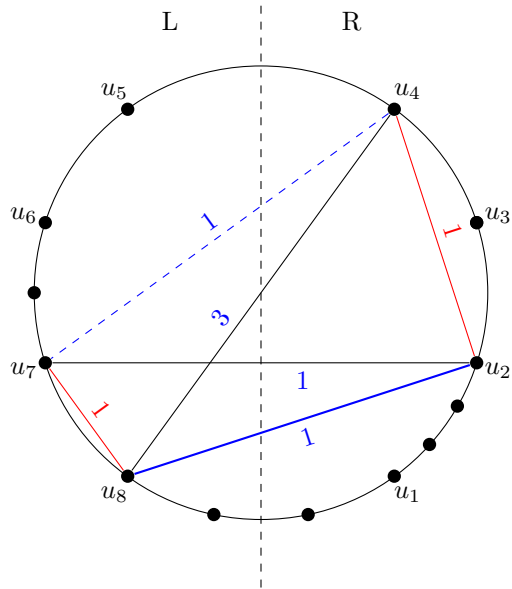


Figure 3: Second iteration, between edges  $u_2u_7$  and  $u_4u_8$ , with minimum capacity 1. The edge  $u_2u_8$  is marked in white.

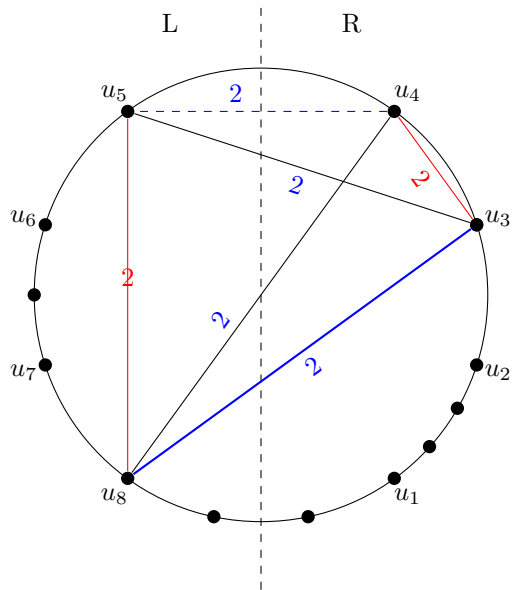


Figure 4: Third iteration, between edges  $u_3u_5$  and  $u_4u_8$ .  $u_3u_8$  becomes white.

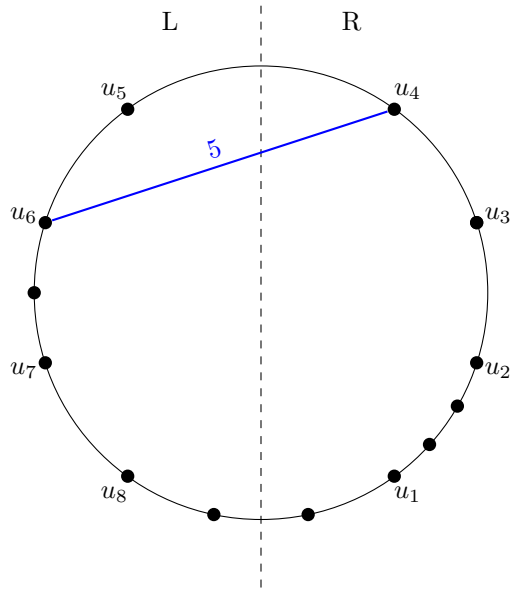


Figure 5: Last iteration, this time  $i = j'$  and  $j = i'$ .  $u_4u_6$  is marked in white.

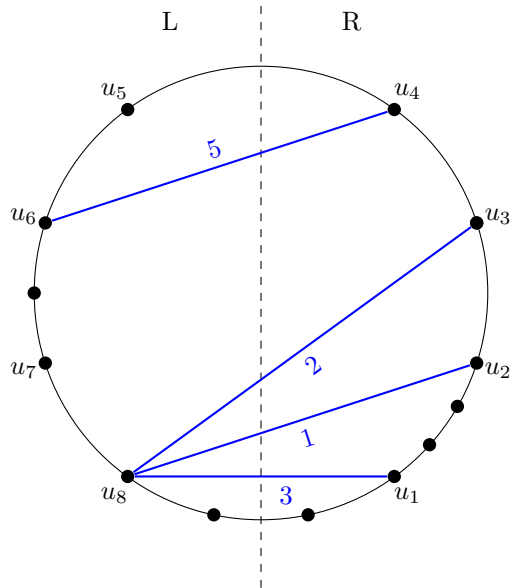


Figure 6: The white edges (in blue) are uncrossed. Their capacities are given by the uncrossing lemma, applied to the selected crossed pairs..

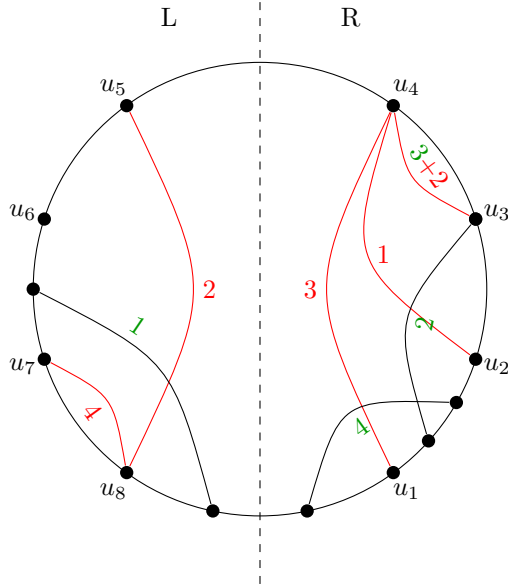


Figure 7: After uncrossing, there is no more bilateral edge.

$2n$  arcs of demand. Then, at the end, we have  $2nc$  paths between the terminals on the boundary of  $F$ . We want to use these paths to route the original demands  $H_F$ .

First, the number of possible planar demand graphs on  $F$  with maximum degree 2 is equal to the number of noncrossing partitions of  $T$ . A *noncrossing partition* of a set  $T = \{t_1, \dots, t_n\}$  is a partition without two parts  $A$  and  $B$ , such that there are  $i < j < k < l$  with  $t_i, t_k \in A$  and  $t_j, t_l \in B$ . The number of noncrossing partitions is well-known to be the  $n$ th Catalan number  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . As we take  $c$  of these graphs, there is at most  $C_n^c$  possible choices of  $2nc$  paths by this method.

Then, let  $\mathcal{P}$  be a set of  $2nc$  paths on  $n$  terminals, each terminal having  $2c$  paths ending at it. We want to glue together paths from  $\mathcal{P}$  in order to get a solution to our original problem. A part will contain an ordering  $P_1, \dots, P_k$  of its paths, where  $P_i$  is a  $(u_i, u_{i+1})$ -path. Such a part satisfies the original demand edge  $(u_1, u_{k+1})$ . Thus, we need to give an upper bound on the number of partitions of  $\mathcal{P}$  in consecutive sub-paths of a path. We can represent  $\mathcal{P}$  as a  $2c$ -regular graph  $H'$  with  $n$  vertices and  $2nc$  edges. We are looking for the number of partition of  $H'$  into paths. But a partition into paths can be encoded in the following way: for each vertex  $v$ , give a matching on  $\delta(v)$ . Two edges are matched if they are consecutive in one of the paths of the partition. An upper bound on the number of partition can then be deduced from an upper bound on the number of matchings in the complete graph with  $2c$  vertices. This last

value is given by

$$\sum_{i=0}^c \binom{2c}{2i} \frac{(2i)!}{2^i} = (2c)! \sum_{i=0}^c \frac{1}{(2(c-i))!} \frac{1}{2^i} \leq c \frac{(2c)!}{2^c} = m_c \quad (1)$$

Then we get an upper bound of  $m_c^n$  possible partitions into paths. It proves that the number of planar or non-planar demand graphs on  $T$  that can be solved by  $c$  applications of Theorem 0.2 is at most  $m_c^n C_n^c$ . But the total number of possible demand graphs is  $\frac{(2n)!}{2^n}$ , and a careful analysis would show that we need  $c \log c = \Omega(\log n)$ .