When Do Gomory-Hu Subtrees Exist?

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Abstract

Gomory-Hu (GH) Trees are a classical sparsification technique for graph connectivity. For an edge-capacitated undirected graph $G = (V, E)$ and subset $Z \subseteq V$ of *terminals*, a GH Tree is an edge-capacitated tree $T = (Z, E(T))$ such that for every $u, v \in Z$, the value of the minimum capacity uv cut in G is the same as in T. It is well-known that there does not always exist a GH tree which is a subgraph (or minor if $Z \neq V$) of G. We characterize those graph-terminal pairs (G, Z) which always admit such a tree. We show that these are the graphs which have no *terminal-$K_{2,3}$ minor*, that is, a $K_{2,3}$ minor whose nodes each corresponds to a terminal. We then show that the pairs (G, Z) which forbid such $K_{2,3}$ terminal-minors arise, roughly speaking, from so-called Okamura-Seymour instances, planar graphs whose outside face contains all terminals. One consequence is a result on cut-sufficient pairs (G, H), that is, multiflow instances where the cut-condition is sufficient to guarantee a multiflow for any capacity/demand weights on G/H. Our results characterize the pairs (G, Z) where G is a graph, $Z \subseteq V(G)$, such that (G, H) is cut-sufficient for any demand graph H on Z.

Keywords. *Graph theory, Gomory-Hu Tree, connectivity, cut condition.*

1 Introduction

The notion of sparsification is ubiquitous in applied mathematics and combinatorial optimization is no exception. For instance, shortest paths to a fixed root node in a graph $G = (V, E)$ are usually stored as a tree directed towards the root. Another classical application is that of Gomory-Hu (GH) Trees [4] which encode all of the minimum cuts of an edge-capacitated undirected graph $G = (V, E)$, with capacities $c : E \to \mathbb{R}^+$. For each $s, t \in V$, we denote by $\lambda(s, t)$ the capacity of a minimum cut separating s and t. Equivalently $\lambda(s, t)$ is the maximum flow that can be sent between s, t in G with the given edge capacities. Gomory and Hu showed that one may encode the $O(n^2)$ minimum cuts by a tree on V.

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A *spanning edge-capacitated tree* for G is a spanning tree $T = (V, E')$ together with a capacity function $c' : E' \to \mathbb{R}^+$. Any edge $e \in E'$ induces a *fundamental cut* $G(L, R)$, where L and R are the node sets of the two components of $T \setminus e$. Here we use $\delta_G(L, R)$ to denote the associated cut in G, that is, $\delta_G(L, R) = \{e \in E(G) : e$ has one endpoint in L and the other in $R\}$, and $c(G(L, R))$ to denote the sum of the capacities of the edges of $\delta_G(L, R)$. We may omit the subscript if the context is clear and also just write $\delta(L)$ and $c(\delta(L))$. The definition holds, however, even for disjoint L, R which do not partition V.

Definition 1.1. *Let T be a spanning edge-capacitated tree. An edge $e = uv \in E(T)$ is* encoding *if its fundamental cut $G(L, R)$ is a minimum uv-cut and its capacity is $c'(e)$, that is, $c(G(L, R)) = c'(e)$.*

A *Gomory-Hu tree* (GH tree for concision) is a spanning edge-capacitated tree all of whose edges are encoding. In this case, it is an exercise to prove that any minimum cut can be found as follows. For $s, t \in V$ we have that $\lambda(s, t) = \min\{c'(e) : e \in T(st)\}$, where $T(st)$ denotes the unique path joining s, t in T.

It is well-known that there may not always exist a GH tree which is a subgraph of G. For instance, every GH tree for the nodes of $K_{3,3}$ is a 5-star (cf. [6] Section 8.6, p. 169). It is a natural question to understand when the existence of such a subtree is possible. One application is to *minimum communication spanning trees*. We are given a capacitated network G which represents a logical network with $c(ij)$ denote the bandwidth of the pipe between nodes i, j. The goal is to find a tree T which minimizes the routing cost $\sum_{ij \in E(G)} c(ij)T(i, j)$, where $T(i, j)$ is the length of the ij-path in T. Hu [5] showed that the optimal tree corresponds to the GH Tree for G. It is natural to seek a tree whose edges are chosen amongst the pairs ij which already have bandwidth set up.

Our first main result characterizes the graphs which admit GH subtrees. More precisely, we say that G has the *GH Property* if for any edge-capacity function $c : E(G) \to \mathbb{R}^+$, G, c has a Gomory-Hu tree T such that T is a subgraph of the edges of G with positive capacity (we cannot use edges with capacity 0). Recall that the 1-*sum* of two disjoint graphs G, H is the graph obtained by identifying a single node $u \in G$ with some node $v \in H$. An *outerplanar graph* is a graph with a planar embedding such that each vertex is on the border of the outer face.

Theorem 1.2. *G has the GH Property if and only if G is the* 1*-sum of outerplanar and K_4 graphs.*

In some applications we only specify a subset $Z \subseteq V$ for which we need cut information. We refer to Z as the *terminals* of the instance. The Gomory-Hu method allows one to store a compressed version of the GH Tree which only captures cut values $\lambda(s, t)$ for $s, t \in Z$, and whose vertices are precisely $V(T) = Z$.

We use Theorem 1.2 to study the generalized version where we are given a graph-terminal pair (G, Z), where G is again endowed with edge capacities $c : E(G) \to \mathbb{R}^+$. A *GH Z-Tree* is a capacitated tree $T = (V(T), E(T))$ (cf. [9] Theorem 15.14, p. 250). Formally, the nodes of T form a partition $\{B(v) : v \in Z\}$ of $V(G)$, with $z \in B(z)$ for each $z \in Z$. The sets $B(v)$ are sometimes called *bags*. Definition 1.1 extends as follows. First, for $X \subseteq V(T)$ we define $B(X) = \cup_{z \in X} B(z)$.

For any adjacent vertices $s, t \in Z$ in T, the *fundamental cut* induced by the edge $e = B(s)B(t)$ of T is then $G(B(L), B(R))$ where L, R are the two components of $T - e$. We then say e is *encoding* if its fundamental cut induces a minimum st-cut in G. As before, if all edges are encoding, then T determines the minimum cuts for all pairs $s, t \in Z$.

Definition 1.3. *Let (G, Z) be a graph-terminal pair, $Z \subseteq V(G)$. A GH Z-tree is a* terminal-minor GH tree *if (i) each bag $B(z)$ induces a connected subgraph of G and (ii) for each $st \in T$, there is an edge of G with one end in $B(s)$ and the other in $B(t)$.*

We characterize those graph-terminal pairs (G, Z) which admit GH terminal minors for any edge capacities on G. Our starting point is the following elementary observation.

Proposition 1.4. *$K_{2,3}$ with unit capacities has no Gomory-Hu tree that is a subgraph of itself.*

Even if a graph-terminal pair (G, Z) admits a terminal-minor GH tree, it may still contain a $K_{2,3}$ minor. For instance, we could choose Z to be any two nodes in a $K_{2,3}$ itself. The proposition implies, however, that it should not have a $K_{2,3}$ minor where all nodes in the minor are terminals. Given a set Z of terminals, we say that a graph H is a *terminal minor* of G if $V(H) \subseteq Z$ and for each $z \in V(H)$, there is a *bag* $B(z) \subseteq V(G)$, such that

(i) each bag $B(z)$ contains its terminal z,

(ii) each bag is connected in G,

(iii) bags are disjoint: for distinct $y, z \in V(H)$, $B(y) \cap B(z)$ is empty,

(iv) if $y, z \in V(H)$ are adjacent in H, there is an edge in $G(B(y), B(z))$.

Notice that the union of the bags need not be $V(G)$ in a Z-minor.

In other words, it is a minor such that each $v \in V(H)$ arises by contracting a connected subgraph which contains a node from Z, and possibly deleting arbitrarily many non-terminal nodes. By Proposition 1.4, a natural necessary condition for the graph-terminal pair (G, Z) to always contain terminal-minor GH trees is that G must not contain a $K_{2,3}$ as a terminal minor. For convenience, for any graph H, we will say that the graph-terminal pair (G, Z) contains a *terminal-H minor* if it contains H as a terminal minor, othersize it is *terminal-H minor free*. We will often consider the set Z of terminals to be implicitely given, and will abusely speak about terminal-H minors of a graph or about terminal-H free graphs, meaning with respect to some fixed set Z.

We show that being terminal-$K_{2,3}$ free is also sufficient for having a terminal-minor GH tree (see Section 5) by building on Theorem 1.2.

Theorem 1.5. *Let $G = (V, E)$ be a graph, and let $Z \subseteq V(G)$. The graph terminal pair (G, Z) admits a terminal-minor GH tree for any capacity function $c : E \to \mathbb{R}^+$ if and only if (G, Z) is terminal-$K_{2,3}$ minor free.*

Establishing the sufficiency requires a better understanding of terminal minor-free graphs. We show that the family of graph-terminal pairs (G, Z) that do not have a terminal-$K_{2,3}$ minor arises precisely as subgraphs of *Z-webs*. Z-webs are built from planar graphs with one outside face which contains all the terminals Z and each inner face is a triangle to which we may add arbitrary subgraphs connected to the three nodes. Subgraphs of Z-webs are called *Extended Okamura-Seymour Instances*. Throughout the paper, we will use the term *2-connected* for 2-vertex-connected.

Theorem 1.6. *Let (G, Z) be a graph-terminal pair where G is 2-connected. Then (G, Z) is terminal-$K_{2,3}$ minor free if and only if either G has at most 4 terminals or it is an Extended Okamura-Seymour Instance.*

This will imply the following corollary.

Corollary 1.7. *A graph-terminal pair (G, z) is terminal-$K_{2,3}$ minor free if and only if for any 2-connected block B of G, the subgraph obtained by contracting every edge not in B is terminal-$K_{2,3}$ minor free.*

These results also yield an interesting consequence for multiflow problems. Let G, H be graphs such that $V(H) \subseteq V(G)$. Call a pair (G, H) *cut-sufficient* if the cut condition is sufficient to characterize the existence of a multiflow for any demands on edges of H and any edge capacities on G. If $Z \subseteq V(G)$, we also call (G, Z) *cut-sufficient* if (G, H) is cut-sufficient for any graph on Z.

Corollary 1.8. *(G, Z) is cut-sufficient if and only if it is terminal-$K_{2,3}$ free.*

The literature contains other results on cut sufficiency. For instance, results of Lomonosov and Seymour ([7, 11], cf. Corollary 72.2a [9]) yield a characterization of the demand graphs H such that every supply graph G "works" for H, i.e., (G, H) is cut-sufficient for *any* graph G with $V(H) \subseteq V(G)$. They prove that any such H is (a subgraph of) either K_4, C_5 or the union of two stars. Another question asks for which (supply) graphs G is it the case that (G, H) is cut-sufficient for every H which is a subgraph of G; Seymour [12] shows that this is precisely the class of K_5 minor-free graphs. We refer the reader to [3] for discussion and conjectures related to cut-sufficiency.

The paper is structured as follows. In the next section we prove that every outerplanar instance has a GH tree which is a subgraph. In Section 3 we present the proof of Theorem 1.2. Section 5 wraps up the proof of Theorem 1.5 using Theorem 1.6 whose proof details are deferred to the appendix. We prove Corollary 1.8 in the final Section 6.

1.1 Some Notation and a Lemma

For every capacitated graph $G = (V, E)$ and a node pair s, t in V, there is a minimum cut $\delta(X)$ which is *central*, a.k.a. a *bond*: that is, $G[X], G[V \setminus X]$ are connected. We also denote such a cut by $G[X, V \setminus X]$ and we call $X, V \setminus X$ the *shores* of the cut. If needed, we use subscripts to explicitly refer to the graph, e.g., $\delta_G(X)$. For any $X \subseteq V(G)$ we use shorthand $c(X)$ to denote the capacity of the cut $\delta(X) = \sum_{e \in \delta(X)} c(e)$. For disjoint sets $X, Y \subseteq V(G)$, $c(X, Y)$ denotes the sum of capacities of all edges with one endpoint in X, and the other in Y. We consistently use $c'(e)$ to denote the computed capacities on edges e in some Gomory-Hu tree.

We always work with connected graphs and sometimes with k-connected graphs, by which we always mean k-vertex-connected graphs. Recall that the *block decomposition* of a graph $G = (V, E)$ is the partition of E into its maximal 2-connected subgraphs.

We also usually assume (without loss of generality) that the edge capacities $c(e)$ have been adjusted so that no two cuts have the same capacity.[1] Indeed, this can only restrict the set of possible GH trees to a single tree that must also be a GH-tree in the original graph (with slightly different capacities). As a consequence of this assumption, the minimum st-cut is unique for any pair of nodes s, t. This also implies that the GH Tree is unique. To see this, let T be a GH tree. Let $e = uw \in E(T)$ an edge of T, consider the fundamental cut $G(U, W)$ defined by e, with $u \in U$, $w \in W$. $G(U, W)$ is the unique minimum uw-cut so it must appears in any GH tree. Thus, any GH tree T' has a unique edge $e' = u'w'$ between U and W, with $u' \in U$, $w' \in W$. Suppose that $u \neq u'$ (say), and consider the unique minimum uu'-cut, let U' be its shore containing u. T contains a path traversing u', u, w, w' in that order, hence we have $U' \cap \{u', u, w, w'\} = \{u'\}$. T' contains a path traversing u, u', w', w in that order, hence we have $U' \cap \{u', u, w, w'\} = \{u', w, w'\}$, contradiction. Hence $u'w' = uw$, the GH tree is unique. We note that the same arguments also work for GH Z-trees:

Proposition 1.9. *Let $G = (V, E)$ be a capacitated graph with terminals $Z \subseteq V$. Suppose that for any pair of terminals u, w, there is a unique minimum uw-cut. Then any two GH Z-trees are isomorphic including the composition of bags.*

If C is a subset of nodes, or a subgraph, we use $N(C)$ (or $N_G(C)$ if explicitly needed) to denote its *neighbour set* $\{v \in V(G) \setminus C : \exists u \in C, uv \in E\}$. Let H be an arbitrary graph. A subgraph R is 3-*separated at* X if $X \subseteq V(R)$, $|X| = 3$, X is an independent set in R and $N_G(R \setminus X) = X$, see Figure 1.1. A subgraph is 3-separated in H if it is 3-separated at some X.

An *OS-instance* (for Okamura-Seymour) is a planar graph where all terminals appear on the boundary of the outer face. An *Extended OS Instance* is obtained from an OS-instance H by adding arbitrary graphs, called 3-*separated graphs*, each connected to up to three nodes of some inner face of the Okamura-Seymour instance. We also require that 3-separated graphs in a common face cannot be crossing each other in that face, meaning that if, for each X such that there is 3-separated subgraph at X, we add a complete graph on X to G, then the resulting graph is still planar. H is called the *planar part* of the instance.

The following lemmas prove that, as long as we are only concerned by minimum cuts between terminals, extended OS instances are not more general than OS instances.

Lemma 1.10. *Let G be an extended OS instance and F be a 3-separated graph whose attachment nodes to the planar part are $\{x, y, z\}$. We can define a new graph G' from G by removing $V(F) \setminus$*

[1] This can be achieved in a standard way by adding multiples of $2^{-\delta}$ where $\delta = O(|E|)$.

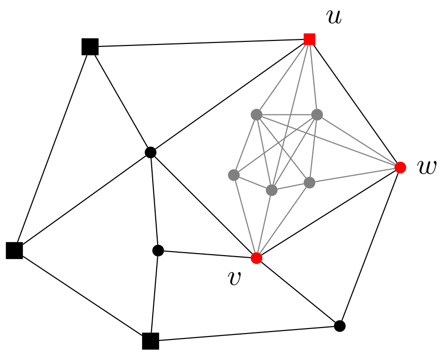

Figure 1: An 3-separated set R (the dashed subgraph) with set $X = \{u, v, w\}$ in red. This graph is an extended OS instance (the square vertices are the terminal), whose planar part contains all the plain black edges.

$\{x, y, z\}$ and add three edges xy, yz, zx with capacities c_{xy}, c_{yz}, c_{zx} so that minimum cuts separating disjoint sets of terminals in Z have the same capacities in G' and in G.

Proof. For each $\alpha \in \{x, y, z\}$, let c_α be the value of a minimum cut in F separating α from $\{x, y, z\} \setminus \{\alpha\}$. We use S_α to denote the shore of such a cut in F, where $\alpha \in S_\alpha$. We replace F in G by a claw where the central node is a new node u_H, and leaves are x, y and z, and the capacity of $u_H\alpha$ is c_α for any $\alpha \in \{x, y, z\}$. We claim that this transformation preserves the values of minimum cuts between sets of terminals.

Notice that $c_\alpha \leq \sum_{\beta \in \{x,y,z\} \setminus \{\alpha\}} c_\alpha$, hence a minimum cut induced by S' in G' where $x \in S'$ but $y, z \notin S'$ will also have $u_H \notin S'$. For such a set S' in G' with $x \in S, u_H, y, z \notin S$, we may then identify a cut with the same capacity in G induced by $S := S' \cup S_x$. Reciprocally, given a cut S of G with $x \in S$, $y, z \notin S$, the cut $S' := S \setminus (V(F) \setminus \{x, y, z\})$ has capacity at most the capacity of S. Thus the values of minimum terminal cuts are preserved.

Finally we can remove u_H and its incident edges and replace them with three edges xy, yz, zx without changing the capacities of the cuts, by posing $c_{xy} := \frac{c_x + c_y - c_z}{2}$, $c_{yz} := \frac{c_y + c_z - c_x}{2}$ and $c_{zx} := \frac{c_z + c_x - c_y}{2}$ (note that these quantities are positive). Indeed this new triangle has the same cut capacities as the tripod of edges incident to u_H.

■

Since the 3-separated graphs are non-crossing, we may iterate the process to obtain the following.

Lemma 1.11. *For any extended OS instance G we can replace each 3-separated graph by a three edges to obtain an equivalent (planar) OS instance G'. It is equivalent in that for any partition $Z_1 \cup Z_2 = Z$, the value of a minimum cut separating Z_1, Z_2 in G is the same as it is in G'.*

As we use the following lemma several times throughout we introduce it now.

Lemma 1.12. *Let $t \in V(G)$ and X, Y be disjoint subsets which induce respectively a minimum xt-cut and a minimum yt-cut where $x \in X, y \in Y$. For any non-empty subset M of V which is disjoint from $X \cup Y \cup \{t\}$, we have $c(M, V \setminus (X \cup Y \cup M)) > 0$.*

Proof. We have

$$\begin{aligned}
&c(M \cup X) + c(M \cup Y) \\
&= c(X) + c(Y) + 2c(M, V \setminus (X \cup Y \cup M)) \\
&< c(M \cup X) + c(M \cup Y) + 2c(M, V \setminus (X \cup Y \cup M))
\end{aligned}$$

where the second inequality follows from the fact that $\delta(M \cup X)$ (respectively $\delta(Y \cup M)$) separates t from X (respectively Y) but $M \cup X \neq X$ (respectively $M \cup Y \neq Y$). ■

The definition of the GH Property for G requires that the desired subtrees exist in any subgraph. The property also holds after contracting an edge e. Indeed GH Trees in the contracted graph are in 1-1 correspondence to GH trees in the original graph where we set $c(e) = \infty$. We obtain a tree in the larger graph by adding a pendant leaf with capacity ∞. Hence:

Proposition 1.13. *The GH Property is closed under taking minors.*

Recall that a 1-*sum* of two disjoint graphs $G = (V, E)$ and $H = (U, F)$ is a graph obtained from the union of G and H by identifying a vertex $v \in V$ with a vertex $u \in U$. The 1-sum operation is also well-behaved relative to GH trees and the GH property. To see this, suppose that G is obtained by the 1-sum of two graphs H_1, H_2 at a node v. Note that for any $s, t \in V(G)$, there is a minimum st-cut $G[A, B]$ which is central. Without loss of generality either $A \subseteq V(H_1) \setminus v$ or $A \subseteq V(H_2) \setminus v$. If the latter holds, then this cut is exactly the same as the cut $\delta_{H_2}(A)$. One may now verify that a GH Tree for G is obtained by taking the union of GH Trees for H_1, H_2. Hence:

Proposition 1.14. *The GH property is closed under 1-sums. Moreover, if a graph G has the GH property, each block of G has the GH property.*

The second part of the proposition follows from the fact that the restriction of a GH tree for G to the subset of vertices of one block will be a GH tree for that block.

2 Outerplanar graphs have Gomory-Hu Subtrees

Theorem 2.1. *Any 2-connected outerplanar graph G has a Gomory-Hu tree that is a subgraph of G.*

Proof. Let G be an outerplanar graph with outer circuit $C = v_1, v_2, \ldots, v_n$ (C exists because G is 2-connected). As discussed in Section 1.1, we assume that no two cuts have the same capacity, so

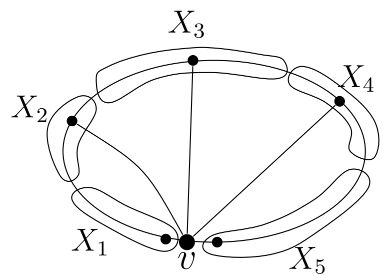

Figure 2: Edges of T into each component of $T \setminus v$ on outer face.

let T be the unique Gomory-Hu tree of G (see Section 1.1). We want to prove that T is a subgraph of G.

Notice that the shore of any min-cut in G is a subpath $v_i, v_{i+1}, \ldots, v_{j-1}, v_j$ (indices taken modulo n) because we assumed for any min-cut $\delta(S)$, both S and $V - S$ induce connected subgraphs.

Let v be any node and consider the fundamental cuts associated with the edges incident to v in the Gomory-Hu tree. The shores (not containing v) of these cuts define a partition $X_1, X_2, \ldots X_k$ of $V \setminus \{v\}$ where each X_i is a subpath of C. We may choose the indices such that $v, X_1, \ldots, X_k$ appear in clockwise order on C – see Figure 2.

Claim 2.2. *For each $i \in \{1, \ldots, k\}$, there is an edge in G from v to some node in X_i.*

Proof. We prove this by contradiction, so assume there is no edge from v to some X_i. Notice that $i \notin \{1, k\}$ because of the edges of C. Let j be the maximum index in $\{1, \ldots, i-1\}$ with $c(v, X_j) \neq \emptyset$, and let $j' \in \{i+1, \ldots, k\}$ minimum with $c(v, X_{j'}) \neq \emptyset$, hence $c(v, M) = \emptyset$ where $M := X_{j+1} \cup X_{j+2} \ldots \cup X_{j'-1}$. By taking $X = X_j, Y = X_{j'}, t = v$, Lemma 1.12 implies that $c(M, V \setminus (X_j \cup X_{j'} \cup M) > 0$. However, outerplanarity and the existence of edges from both X_j and $X_{j'}$ to v, imply that there is an edge between v and M, see Figure 3. This contradicts the choice of i, j or j'. ∎

Let $xy \in E(T)$ be an edge of the Gomory-Hu tree. We must prove that $xy \in E(G)$. Let $\delta(X)$ be the fundamental cut associated with xy, with $x \in X$, define $Y = V \setminus X$. As in the preceding arguments we may use the fundamental cuts associated with edges incident to x and partition $X \setminus \{x\}$ into min-cut shores $X_1, X_2, \ldots, X_k$; we do this by ignoring the one shore Y. Similarly, we may partition $Y \setminus \{y\}$ into min-cut shores $Y_1, Y_2, \ldots, Y_l$. We can label these so that $X_1, X_2, \ldots, X_k, Y_1, \ldots, Y_l$ appear in clockwise order around C - see Figure 4. There are two cases for the position of x. Either there is some $i \in \{1, \ldots, k-1\}$ such that x is between X_i and X_{i+1} or x lies on the "fringe", i.e., it is adjacent to some node of Y by an edge of C. Similarly, either y lies on the fringe or there exists $j \in \{1, \ldots, l\}$ such that y is between Y_j and Y_{j+1}. In the fringe cases,

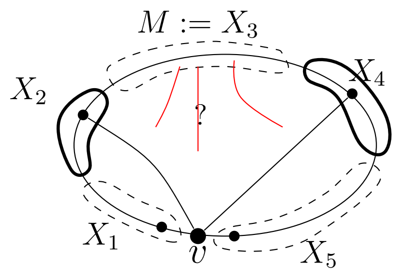

Figure 3: Configuration occurring in the proof of Claim 1.

either $xy \in C$ or the argument is similar to (and easier than) the non-fringe case so we focus on them.

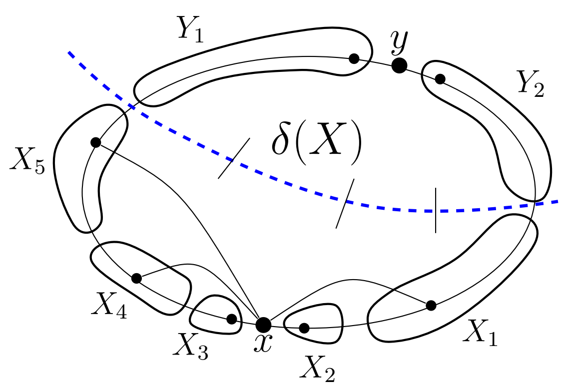

Figure 4: An arbitrary edge $xy \in T$.

By contradiction suppose $xy \notin E(G)$. By Claim 2.2, there is an edge e from x to Y, let $m \in \{1, \dots, l\}$ such that $e \in \delta(x, Y_m)$. If $m \notin \{1, l\}$, by outerplanarity either $\delta(y, Y_1)$ or $\delta(y, Y_l)$ is empty; this contradicts Claim 2.2. By symmetry we may assume $e \in \delta(x, Y_1)$. By a similar argument there is an edge $e' \in \delta(y, X_1)$. By Claim 2.2, there are also two edges $e'' \in c(x, X_1)$ and $e''' \in c(y, Y_1)$.

Let $X' = \{x\} \cup X_2 \cup \ldots \cup X_k$ and $Y' = \{y\} \cup Y_2 \cup \ldots \cup Y_l$, $\delta(X')$ is a cut separating x from X_1 and similarly $\delta(Y')$ separates y from Y_1. As $\delta(X_1)$ is the fundamental cut between x and X_1, we have that $c(X_1) < c(X')$, and similarly $c(Y_1) < c(Y')$. Now, because of the edges e, e', e'', e''', by outerplanarity there is no edge between X' and Y', hence

$$c(X_1) + c(Y_1) = c(X') + c(Y') + 2c(X_1, Y_1) > c(X_1) + c(Y_1) + 2c(X_1, Y_1)$$

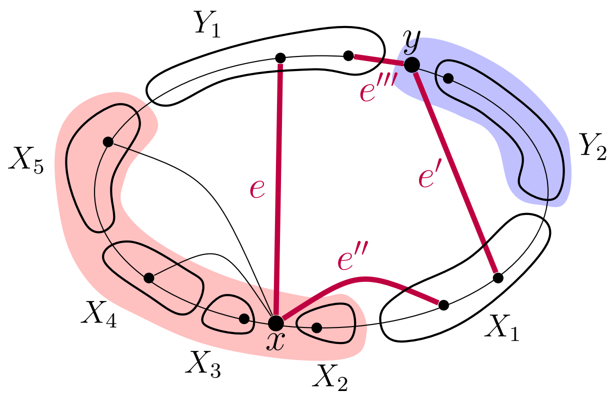

Figure 5: Showing that $xy \in T$ must be an edge of G.

a contradiction. ■

3 Which Instances have Gomory-Hu Subtrees?

The previous result leads to a characterization of graphs with the *GH Property*: that is, graphs whose capacitated subgraphs always contain a Gomory-Hu Tree as a subtree. We prove this characterization in this section.

We start with a simple observation that $K_{2,3}$ does not have a GH subtree.

Proposition 1.4. *$K_{2,3}$, when all edges have capacity 1, has no Gomory-Hu tree that is a subgraph of itself.*

Proof. Let $\{u_1, u_2\}, \{v_1, v_2, v_3\}$ be the bipartition. Since the minimum u_1, u_2 cut is of size 3, a GH tree should contain a u_1u_2-path all of whose edges have capacity at least 3. If this path is $u_1v_1u_2$, then the tree's fundamental cut associated with u_1v_1 must be a minimum u_1v_1-cut. But this is impossible since $\delta(v_1)$ is a cut of size 2. ■

This leads to the desired characterization.

Theorem 1.2. *G has the GH Property if and only if G is the 1-sum of outerplanar and K_4 graphs.*

Proof. First suppose that G is such a 1-sum. Consider the block decomposition of G. Each outerplanar block in this sum has the GH Property by Theorem 2.1. So consider a K_4 block and a subgraph G' of K_4 with edge capacities. If G' is K_4, then clearly any GH tree is a subtree. Otherwise G' is a proper subgraph of K_4 and hence is outerplanar. It follows that each block has the GH Property. Because the GH property is closed by 1-sum, G has the GH property.

For the reverse direction, suppose that G has the GH property. By Proposition 1.14, each block of G has the GH property. Hence we may assume that G is 2-connected, and prove that it is either K_4 or an outerplanar graph.

Because the GH property is closed by minor operations, G has no $K_{2,3}$ minor. Outerplanar graphs are graphs with forbidden minors $K_{2,3}$ and K_4, see [9] p. 28. Hence if G is not outerplanar, then it has a K_4 minor. Notice that any proper subdivision of K_4 contains a $K_{2,3}$ minor, as well as any graph built from K_4 by adding a path between two distinct nodes contains a $K_{2,3}$ minor. Hence G must be K_4 itself. The result now follows. ■

In Section 5, we extend this result to the case where a subset of terminals is specified.

4 Characterization of terminal-$K_{2,3}$ free graph-terminal pairs

In this section we prove Theorem 1.6. Throughout, we assume that we have an undirected 2-connected graph G with terminals $Z \subseteq V(G)$.

We first check sufficiency of the condition of Theorem 1.6. Any graph with at most 4 terminals is automatically terminal-$K_{2,3}$ free and one easily checks that any extended Okamura-Seymour instance cannot contain a terminal-$K_{2,3}$ minor. Hence we focus on proving the other direction: any terminal-$K_{2,3}$ minor-free 2-connected graph G lies in the desired class. To this end, we assume that $|Z| \geq 5$ and we ultimately derive that G must be an extended OS instance.

We start by excluding the existence of certain K_4 minors.

Proposition 4.1. *If $|Z| \geq 5$ and G has a terminal-K_4 minor, then G has a terminal-$K_{2,3}$ minor.*

Proof. Let K_4^+ be the graph obtained from K_4 by subdividing one of its edges. By removing the edge opposite to the subdivided edge, we see that K_4^+ contains $K_{2,3}$. Hence it suffices to prove that G contains a terminal-K_4^+ minor.

Consider a terminal-K_4 minor on terminals $Z' = \{s, t, u, v\}$. We may assume this is obtained from contracting node-disjoint trees T_x for each terminal $x \in Z'$, such that for any $x, y \in Z'$, there is an edge e_{xy} having one extremity in T_x and one in T_y. We may assume that $T_x = \bigcup_{y \in Z' \setminus \{x\}} P[x, y]$, where $P[x, y]$ is a path in G from x to an end of e_{xy} and not containing e_{xy}, see Figure 4, left side. Denote $U := \bigcup_{x \in Z'} V(T_x)$.

As $|Z| \geq 5 > |Z'|$, there is some terminal $w' \notin Z'$. If $w' \notin U$, then let Q be a minimal path which joins w' to some $w \in U$; otherwise let $w = w'$ and Q be this singleton. Without loss of generality, w is in T_s. Suppose first that w lies in exactly one of the paths $P[s, u]$, $P[s, v]$, $P[s, t]$, say $P[s, u]$. We then obtain a terminal-K_4^+ minor by contracting Q into a single node q. This node q is a terminal because Q contains the terminal w', and q is the terminal which subdivides the minor edge su.

Consider next the case where w lies in exactly 2 of the paths, say $P[s, u], P[s, v]$. In this case, we obtain the desired minor after contracting Q again in a terminal q, by exchanging the role of s and q, so that q is a degree 3 node of the K_4 minor. Hence s can play the role of the degree 2 node in a terminal-K_4^+ minor, as desired.

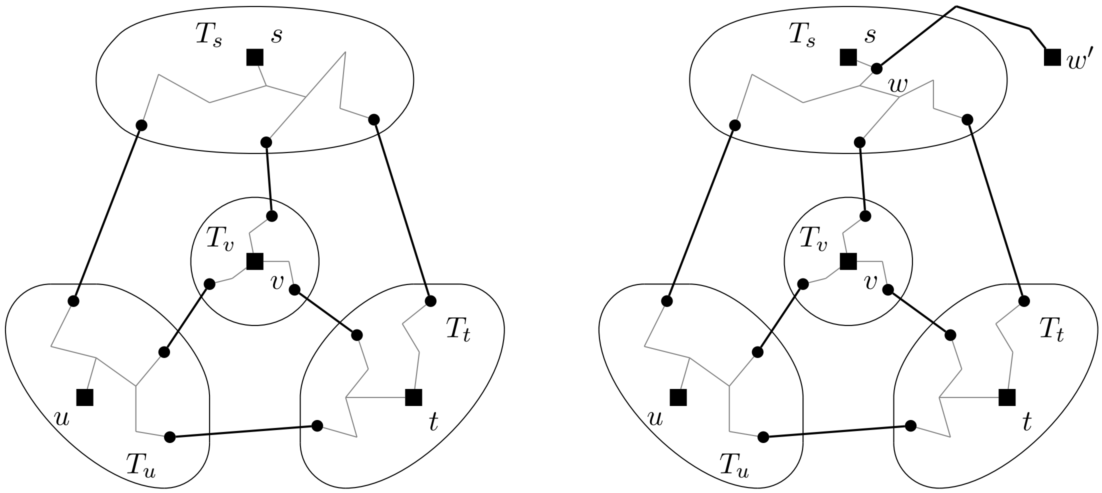

Figure 6: An illustration of a terminal-K_4 minor embedded in a graph. On the right, a *bad case*: w' is connected to T_s on the intersection of the three paths to each other terminal.

In the last case, w lies in all three of the paths $P[s,u]$, $P[s,v]$, $P[s,t]$. We call this the *bad case* as we must take more care in selecting Q. Let $R := P[s,u] \cap P[s,v] \cap P[s,t]$ and we can assume that $w' \notin R$. If it were, then we could just replace s by w' and consider s as our "outside" terminal.

By the 2-connectivity of G, there are two vertex-disjoint paths Q_1, Q_2 from w' to U such that $Q_i \cap U = \{w_i\}$, where w_1, w_2 are distinct. If either $w_i \in T_s \setminus R$, then we may use Q_i to be in one of the good cases. If w_1 is in T_x and w_2 is in T_y with $x \neq y$, we can create a K_4^+ minor where w' is the degree 2 terminal for the edge xy.

So we now assume w_1, w_2 are both in R and in particular $|R| \geq 2$. Let z be the endpoint of R which is not s. Define $U' := U \setminus (V(R) \setminus z)$ and note that s, w' are in the same component, K, of $G' = G \setminus U'$. There exists edges zz', bb' with distinct endpoints such that $z', b' \in K$ and $b \in U'$. Indeed, the 2-connectivity of G implies that no vertex covers every edge in $\delta(U', K)$, hence there are two disjoint edges in $\delta(U', K)$. If none of these two edges contains z, we can replace one of them by an edge in $\delta(z, k')$.

By the 2-connectivity of G, there exist vertex-disjoint paths $R_s, R_{w'}$ in K from $\{s, w'\}$ to $\{z', b'\}$. Without loss of generality, we may contract s into z' and w' into b' using these two paths (at this point, we do not care which terminal is which). Moreover, since K is connected we may contracted edges to make a minor of K where sw' is an edge. Using this construction, we are back in some of the earlier cases: either b lies in one of the paths $P[s,x] \setminus R$, or $b \in T_x$ for some $x \neq s$. In both cases, we create a K_4^+ minor as before.

■

Now we have ruled out the existence of terminal-K_4 minors, we start building up minors which can be possible.

Proposition 4.2. *Any 2-connected graph with terminals Z, with $|Z| \geq 3$, has a 2-connected minor H with $V(H) = Z$.*

Proof. Clearly there is a 2-connected minor H with $V(H) \supseteq Z$. Choose one which minimizes $|V(H)|$ and suppose there is a non-terminal node in H. In particular we may assume there is an edge sv with $s \in Z$, $v \notin Z$. By minimality, contracting sv decreases the connectivity to 1. Hence, $\{s, v\}$ is a cut separating two nodes t and t'. Thus, there are two disjoint tt'-paths, one containing s and the other v. That is, there is a circuit C containing s, t, v, t' in that order.

By minimality of H, we also have that $H - sv$ is not 2-connected. It follows that $H - sv$ contains a cut node $\{z\}$ where s, v lie in distinct components of $H - sv - z$. This would contradict the existence of C, and this completes the proof. ■

As $|Z| \geq 5$, the previous proposition implies that there is a terminal-C_4 minor. We now show that G contains a terminal-C_k minor where $k = |Z|$.

Proposition 4.3. *Consider a 2-connected graph G with terminals Z such that (G, Z) is terminal-$K_{2,3}$ minor free, and let k be maximum such that G contains a terminal-C_k minor. Then $k = |Z|$.*

Proof. By Proposition 4.2, let H be a 2-connected terminal-minor of G with $V(H) = Z$. Consider an ear-decomposition of H, starting with longest cycle C_0 and ears $P_1, \ldots, P_k$. Then all ears are single edges (from which the proposition follows), otherwise let P_i be an ear that is not a single edge, with i minimum. The two ends of P_i are nodes x, y of C_0. If x and y are consecutive in C_0, this contradicts the maximality of C_0. If they are not consecutive, $C_0 \cup P_i$ is a subdivision of $K_{2,3}$. ■

We let $k = |Z|$ henceforth. A terminal-C_k minor of G can also be represented as a collection of k node-disjoint subtrees $T_1, \ldots, T_k$, where each T_i contains exactly one terminal t_i. There also exist edges $e_1, \ldots, e_k$, where e_i has one extremity u_i in T_i and the other, v_{i+1}, in T_{i+1}. The subscript $k + 1$ is taken to be 1; the edges in the subtrees are the contracted edges and the edges $e_1, \ldots, e_k$ are the undeleted edges. We define s_i as the only node in $V(P[t_i, u_i]) \cap V(P[u_i, v_i]) \cap V(P[v_i, t_i])$, where $V(P[x, y])$ is the node set of the path with ends x and y in the tree T_i. Thus, T_i is $P[s_i, u_i] \cup P[s_i, v_i] \cup P[s_i, t_i]$.

We denote by S_i the path from t_i to s_i in T_i and we take our representation so that $\sum_{i=1}^{k} |S_i|$ minimized. We denote by P_i the path from s_i to s_{i+1} in $T_i \cup \{e_i\} \cup T_{i+1}$, that is $P_i := P[s_i, u_i] \cup \{e_i\} \cup P[v_{i+1}, s_{i+1}]$.

Proposition 4.4. $\sum_{i=1}^{k} |S_i| = 0$.

Proof. By contradiction, suppose $|S_1| > 0$ and so t_1 does not lie in the graph induced by $D = P_1 \cup \ldots \cup P_k \cup S_2 \cup \ldots \cup S_k$. By 2-connectivity, there are two vertex-disjoint paths in G from t_1 to distinct nodes x and y in D. Moreover we may choose that $x = s_1$. To see this, suppose that

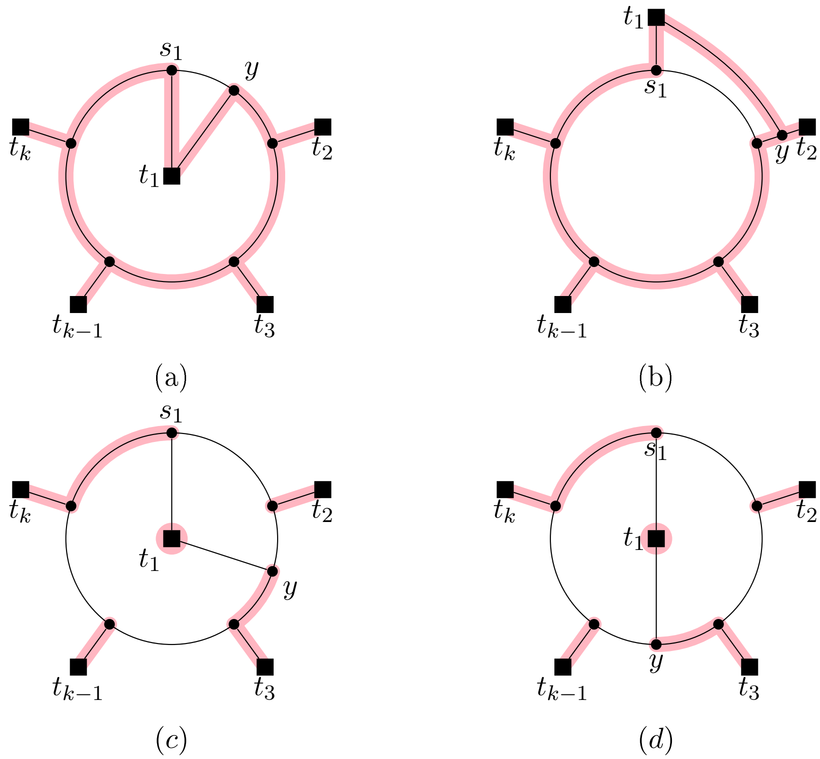

Figure 7: In cases (a) and (b), we reducing $|S_1|$ by keeping the shaded subgraphs. In cases (c) and (d) the shaded edges are contracted to get a terminal-$K_{2,3}$ minor.

$z \in S_1$ is the closest node to s_1 which is used by the one of the paths (possibly $z = t_1$). We may then re-route one of the paths to use the subpath of S_1 from z to s_1.

If y is contained in one of P_k, P_1, it is routine to get another representation of the minor where all the S_i are at least as short, and S_1 is empty, contradicting the minimality of our choice of representation. A similar argument holds if $y \in S_k \cup S_2$, see Figure 7 cases (a) and (b).

So we assume $y \in D \setminus (P_k \cup P_1 \cup S_k \cup S_2)$. We now find a terminal-$K_{2,3}$ minor, and that is again a contradiction. To see this, let T_i be a tree which contains the second node y. As $k \geq 5$, we may assume either $i \in [4, k]$, or $i \in [2, k-2]$. Suppose the latter as the two cases are similar. Note that if $i = 2$, then y is in $P[s_2, u_2]$. Then we obtain a terminal-$K_{2,3}$ where the two degree-3 nodes correspond to the terminals in T_i and T_k. The degree-2 nodes will correspond to t_1, t_2 and t_{k-1} — see Figure 7 cases (c) and (d). ■

Hence there is a circuit C in G containing every terminal. Let the terminal in C in cyclic order be t_1, t_2, $\ldots t_k$.

Proposition 4.5. *There are no two node-disjoint paths in G, one from t_i to $t_{i'}$, the other from t_j to $t_{j'}$, with $i < j < i' < j'$.*

Proof. For convenience, let's denote $s = t_i$, $t = t_{i'}$, $s' = t_j$ and $t' = t_{j'}$. We will prove by contradiction. Let P be the st-path and Q the $s't'$-path in G. We may assume that we choose P and Q to minimize their total number of maximal subpaths disjoint from C.

We consider the set (not multi-set) of edges $E(C) \cup E(P) \cup E(Q)$, and only keep s, s', t, t' as terminals. This defines a subgraph G' of G of maximum degree 4 by construction. Contract edges in $E(C) \cap (E(P) \cup E(Q))$, and then contract edges so that nodes of degree 2 are eliminated. This gives a minor H where the only nodes not of degree 4 are s, t, s', t', which have degree 3. $E(H) \cap E(P)$ induces an st-path P' in H, $E(H) \cap E(Q)$ induces an $s't'$-path Q' in H. P' and Q' are again node-disjoint. We call the remaining edges of $E(C)$ in H *C-edges*. They induce a cycle which alternates between nodes of P' and Q'. To see this, suppose that e is such an edge joining $x, y \in V(P')$ (the case for Q' is the same). We could then replace the subpath of P between x, y by the subpath of C which was contracted to form e. This would reduce, by at least 1, the number of maximal subpaths of P disjoint from C, a contradiction.

Consider the two nodes u' and v' of Q' adjacent to s in H, such that s', u', v', t' appear in that order on Q'. u' and v' each has one more incident C-edge, whose extremities (respectively) are u, v and must then be on $V(P') \setminus \{s\}$. We create a terminal-K_4 minor on s, s', t, t' as follows — see Figure 8, where u, v may be in either order on P'. We contract all the edges of P' except the one e_s incident to s, and all the edges of Q' except the one $e_{u'}$ incident to u' in the direction of t', we get a terminal-K_4 minor with the edges su', sv', uu', vv', e_s and $e_{u'}$. One easily checks that this leads to the desired terminal-K_4 minor. This contradiction completes the proof. ∎

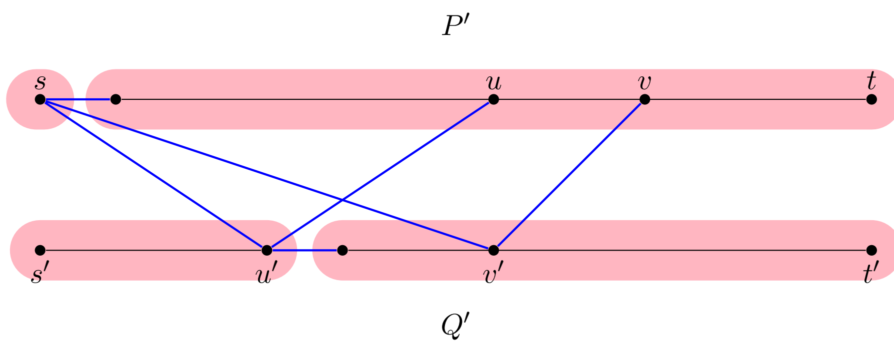

Figure 8: How to get a terminal-K_4 minor: red parts are contracted into single nodes, the blue edges will then form a K_4.

To conclude the characterization of terminal-$K_{2,3}$ minor free graphs, we use (a generalization of) the celebrated 2-linkage theorem. Take a planar graph H, whose outer face boundary is the cycle $t_1, t_2, \ldots, t_k$, and whose inner faces are triangles. For each inner triangle, add a new clique of arbitrary size, and connect each node of the clique to the nodes of the triangle. Any graph built this way is called a $(t_1, \ldots, t_k)$-*web*, or a $\{t_1, \ldots, t_k\}$-web if we do not specify the ordering. Extended OS instances are precisely the subgraphs of Z-webs (the former is less constrained than the latter as we do not ask to start from a graph whose inner faces are triangle, and we allow to glue any 3-separated subgraph instead of only those built from a complete graph).

Theorem 4.6 (Seymour [10], Shiloach [13], Thomassen [14])**.** *Let G be a graph, and $s_1, \ldots, s_k \in V(G)$. Suppose there are no two disjoint paths, one with extremity s_i and $s_{i'}$, and one with extremity s_j and $s_{j'}$, with $i < j < i' < j'$. Then G is the subgraph of an $(s_1, s_2, \ldots, s_k)$-web.*

The linkage theorem is usually stated in the special case when $k = 4$, but the extension presented here is folklore. One can reduce the general case to the case of $k = 4$ by identifying the nodes $s_1, \ldots s_k$ with every other inner node of a ring grid with 7 circular layers and $2k$ rays, and choosing 4 nodes of the outer layer, labelling them s, t, s', t' in this order, and connecting them in a square — see Figure 9. It is easy to prove that there are two node-disjoint paths, one with extremity s and s', the other with extremities t and t' in the graph built this way if and only there are two disjoint paths as in the theorem in the original graph (for instance, use the middle layer to route the path from s to s_i with only 2 bends, then the remaining graph is a sufficiently large subgrid to route the three other paths). Because the grid is 3-connected, its embedding is unique and we get that G is embedded inside the inner layer of the ring, from which the general version of the theorem is deduced.

By using Theorem 4.6 with Proposition 4.5, we get that any 2-connected terminal-$K_{2,3}$ free graph is a subgraph of a Z-web where Z is the set of terminals.

This now completes the proof of Theorem 1.6. □

We now establish Corollary 1.7.

Proof. If G is terminal-$K_{2,3}$ minor-free, then clearly contacting all blocks but one must create a terminal-$K_{2,3}$ free instance.

Conversely, suppose that G has a terminal-$K_{2,3}$ minor. Since this minor is 2-connected, it must be a minor of a graph obtained by contracting or deleting all the edges of every 2-connected component except one. Let's call that last block B. Hence the terminal-$K_{2,3}$ minor is a minor of the graph obtained by contracting all the edges not in B. ■

5 General Case: Gomory-Hu Terminal Trees in terminal-$K_{2,3}$ minor free graphs.

In this section we prove Theorem 1.5 using the characterization of terminal-$K_{2,3}$ minor free graphs. The high level idea is a reduction to Theorem 2.1 by contracting away the non-terminal nodes in the graph.

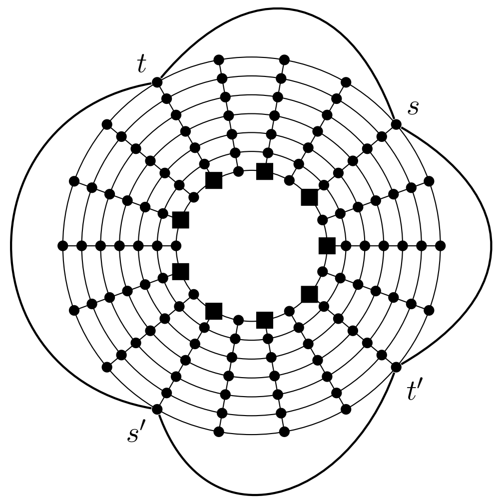

Figure 9: Gadget for the proof of the general linkage theorem, the square notes are $s_1, \ldots, s_k$ and the original graph must lie in the central face of the gadget.

In the following we let (G, Z) denote a connected graph G and terminals $Z \subseteq V(G)$. In this section we study graphs which always have GH Z-trees which are terminal-minor GH trees - see Definition 1.3. We also call these *bag minors*. We say that T occurs as a *weak bag minor* if it occurs as a bag minor in the graph obtained by deleting some non-terminal nodes (so some bags get smaller).

Definition 5.1. *The pair* (G, Z) *has the* GH Minor Property *if for all subgraph* G' *with positive capacities* c'*, there is a GH* Z*-Tree which occurs as a bag minor in* G'*. The pair* (G, Z) *has the* weak *GH Minor Property if for all subgraph* G' *with positive capacities* c'*, there is a GH* Z*-Tree which occurs as a weak bag minor in* G'*.*

An example where we have the weak but not the (strong) property is for $K_{2,3}$ where Z consists of the degree 2 nodes and one of the degree 3 nodes, call it t, and call s the other degree 3 node. Clearly this is terminal-$K_{2,3}$ minor free since it only has 4 terminals. The unique GH Z-tree T is obtained from G by deleting the non-terminal node and assigning capacity 2 to all edges in the 3-star. Hence T is obtained as a minor (in fact a subgraph) of G. However, the bag $B(t)$ consists of the 2 degree-3 nodes s and t, which do not induce a connected subgraph. Indeed s cannot be put in another bag, as otherwise such a bag would induce a cut of capacity 3 whereas the minimum capacity of a cut between a degree-2 node and any other node is 2. Hence T does not occur as a bag minor. Fortunately, such instances are isolated and arise primarily due to instances with at most 4 terminals. We handle these separately.

Proposition 5.2. *Let G be an undirected, connected graph and Z be a subset of at most 4 terminals. If no two central cuts have the same capacity, then the unique GH Z-Tree T occurs as a weak bag minor. Moreover, if T is a path, then it occurs as a bag minor.*

We defer the proof of this and the following lemma to an appendix.

Lemma 5.3. *Let T be a GH Z-Tree bag minor for some graph-terminal pair (G, Z) and let $v \in Z$. Let uv be the edge of T incident to v of maximum weight. If we set $B'(u) = B(v) \cup B(u)$ and $B'(x) = B(x)$ for each $x \in Z \setminus \{u, v\}$, then the resulting partition defines a GH $(Z \setminus v)$-Tree T' which is a bag minor.*

We now prove the following strengthening of Theorem 1.5.

Theorem 5.4. *thm:minorGHstrength Let G be an undirected graph and $Z \subseteq V$. (G, Z) has the weak GH Minor Property if and only if (G, Z) is a terminal-$K_{2,3}$ minor free graph. Moreover, if none of G's blocks is itself a 4-terminal instance, then (G, Z) has the GH Minor Property.*

Proof. If G has a terminal-$K_{2,3}$ minor H, then consider setting capacities as follows. If an edge was deleted to produce the minor H, we set its capacity to 0. If an edge was contracted its capacity is ∞. The remaining edges have capacity 1. It is clear that minimum cuts for this instance correspond to cuts within the $K_{2,3}$ minor itself. Starting from the GH Z-tree T of G, we can also contract the edges of T with infinite capacity, to get a GH tree T' of $K_{2,3}$.

T' must have an edge e between its two degree-3 vertices, where e is not an edge in H. This implies that T also contains e. For T to be a terminal-minor GH subtree, it is necessary that e is an edge of G with positive capacity. But if it has capacity 1, it should be in H, and if it has capacity ∞, the two degree-3 vertices of H should have been identified.

We now consider the converse direction and hence assume that G is a terminal-$K_{2,3}$ minor free graph. Let G' be some subgraph of G with edge capacities $c(e) > 0$, perturbed so that all minimum cuts are unique. We show that the unique GH Z-tree of G' occurs as a (possibly weak) bag minor.

We deal first with the case where G' has cut nodes. Note that one may iteratively remove any leaf blocks which do not contain terminals. This operation essentially does not impact the GH Z-Tree. Now consider any block L. Let G'_L be the minor obtained by contracting the edges of every block except L. By our assumption on leaf blocks, each cut point of L is a terminal in G'_L as it is contracted with at least one other block.

Since this minor G'_L is $K_{2,3}$-free, let Z_L be the set of terminals in G'_L, we show the desired bag minor exists for (G'_L, Z_L). This is sufficient since we can later retrieve the desired bag minor for the original terminal set, as Lemma 5.3 tells us that the partition into bags for (G', Z) is a refinement of the partition into bags for G'_L, Z_L. One checks that a GH Z-Tree for G' is obtained by gluing together the appropriate GH terminal trees in each block. Moreover, since each cut node is a terminal, if each block's tree is a bag minor (resp. weak bag minor), then the whole tree is a bag minor (resp. weak bag minor). Therefore it is now sufficient to prove the result in the case where G' is 2-connected.

If G' has at most 4 terminals, then Proposition 5.2 asserts that it has a weak bag minor for a GH tree. Moreover, if it has less than 4 terminals, then its GH Tree is a path and hence occurs as a bag minor. So we now assume that G' contains at least 5 terminals and hence it is an extended OS instance whose outside face is a simple cycle, by Theorem 1.6. Lemma 1.11 implies that we can replace each 3-separated set by three edges and the resulting graph is planar and has the same pairwise connectivities amongst nodes in Z. It is easy to check that any Z-tree bag minor in this new graph is also such a minor in the original instance. Therefore, it is sufficient to show that any planar OS instance with terminals on the outside face has the desired GH tree ("strong") bag minor.

Denote by $t_1, t_2, \ldots, t_{|T|}$ the terminals in the order in which they appear on the boundary of the outer face. Let $\{B(t) : t \in Z\}$ be the bags associated with the (necessarily unique) GH Z-tree T. We show that (i) each $G'[B(t)]$ is connected and (ii) for any $st \in T$, there is some edge of G' between $B(s)$ and $B(t)$.

Consider the fundamental cuts associated with edges incident to some terminal t. Let X_1,X_2, $\ldots X_k$ be their shores which do not contain t. Since any min-cut is central, each X_i intersects the outside face in a subpath of its boundary. Hence, similar to Claim 2.2 (cf. Figure 2), we can order them X_1,$\ldots$,X_k in clockwise order on the boundary with t between X_k and X_1.

The next two claims complete the proof of the theorem.

Claim 5.5. *For each terminal t, $G'[B(t)]$ is connected.*

Proof. By contradiction, suppose that $G'[B(t)] = G' \setminus (X_1 \cup \ldots \cup X_k)$ has more than one component. Note first that the component containing t must contain a subpath of the outside face which, together with the X_i's, includes all nodes on the outside face. Now let K be a component of $G'[B(t)]$ which doesn't contain t. If $N(K) \subseteq X_i$ for some $i \in \{1, \ldots, k\}$, then $\delta(K \cup X_i)$ is a cut separating t from any node in X_i with capacity strictly smaller than $\delta(X_i)$ (as we assumed no two cuts have the same capacity). This contradicts that X_i induces a minimum st-cut for some $s \in X_i$.

Suppose now that there are $1 \leq j < j' \leq k$ such that $N(K) \cap X_j \neq \emptyset$ and $N(K) \cap X_{j'} \neq \emptyset$, *i.e.* K adjacent to X_j and $X_{j'}$. Choose j minimal and j' maximal. Then one can define a circuit D which traverses C from X_j to $X_{j'}$, and then traverses K and terminates at X_j.

Let M be the union of K, X_j, $\ldots$, $X_{j'}$ and all the components inside D, and $M' = M \setminus (X_j \cup X_{j'})$. By Lemma 1.12 applied to $X = X_j$, $Y = X_{j'}$, we have $c(M', V \setminus M) > 0$. But an edge from M' to $V \setminus M$ would either contradict the planarity (if it has one end in $X_{j+1} \cup \ldots \cup X_{j'-1}$,), the fact that K is a component (if it is between K and $G'[B(t)]$) or the choice of j and j' (if it has one end in K and the other in X_i with $i < j$ or $i > j'$).

■

Claim 5.6. *For each $i \in \{1, \ldots, k\}$, there is an edge from a node in $B(t)$ to a node in X_i.*

Proof. By contradiction, suppose $\delta(B(t), X_i) = \emptyset$, for some $i \in \{1, \ldots, k\}$. Let j maximum and j' minimum such that $j < i < j'$, $\delta(B(t), X_j) \neq \emptyset$ and $\delta(B(t), X_{j'}) \neq \emptyset$. Note that j and j' are defined because X_1 and X_k are adjacent to $B(t)$ by the outer cycle. If we define $M := X_{j+1} \ldots \cup X_{j'-1}$, then $c(M, V \setminus (M \cup X_j \cup X_{j'})) = 0$, contradicting Lemma 1.12 where we take $X = X_j, Y = X_{j'}$. ■

■

6 A Consequence for Multiflows

Recall from the introduction that for a graph G and $Z \subseteq V(G)$, we call (G, Z) cut-sufficient if for any multi-flow instance (capacities on G, demands between terminals in Z), we have feasibility if and only if the cut condition holds.

Let (G, H) be a pair of graphs over the same set of vertices with capacities $c : E(G) \to \mathbb{R}^+$ and demands $d : E(H) \to \mathbb{R}^+$. The *flow-cut* gap of (G, H) is the minimal value α over all c and d such that, if the capacity $c(\delta_G(X))$ of any cut of G is at least α times the value of flow demand $d(\delta_H(X))$ in H across that cut, then the flow demand (H, d) is routable in (G, c). The sufficiency of the cut condition for (G, Z) is equivalent to saying that the flow-cut gap of $(G, K_{|Z|})$ is equal to 1.

Corollary 1.8. *(G, Z) is cut-sufficient if and only if it is terminal-$K_{2,3}$ free.*

Proof. First, if there is a terminal-$K_{2,3}$ minor then we obtain a "bad" multiflow instance as follows. For each deleted edge we assign it a capacity of 0. For each contracted edge we assign it a capacity of ∞. The remaining 6 edges have unit capacity. We now define four unit demands. One between the two degree-3 nodes of the terminal minor and a triangle on the remaining three nodes. It is well-known that this instance has a flow-cut gap of $\frac{4}{3}$ cf. [2, 1].

Now suppose that G is terminal-$K_{2,3}$ free and consider a multiflow instance with demands on Z. By Lemma 1.11, we can replace each 3-separated graph by a degree-3 node and this new OS instance will satisfy the cut condition if the old one did. Hence the Okamura-Seymour Theorem [8] yields a half-integral multiflow in the new instance.

We now show that the flow in the modified instance can be mapped back to the original extended OS instance. We do this one 3-separated graph at a time. Consider the total flow on paths that use the new edges through s obtained via the reduction. Let $d(xy), d(yz), d(zx)$ be these values. We claim that these can be routed in the original F. First, it is easy to see that this instance on F satisfies the cut condition: indeed any violated cut $\delta_F(S)$ would contain exactly one of x, y, z, say x. Hence this cut would have capacity less than $d(xy) + d(xz)$ but since this flow routed through s, this value must be at most c_x which is a contradiction. Finally, the cut condition is sufficient to guarantee a multiflow in any graph if demands only arise on the edges of K_4, cf. Corollary 72.2a [9]. Hence we can produce the desired flow paths in F. ■

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A Proof of Proposition 5.2 and Lemma 5.3

We start with a proof of Proposition 5.2.

Proposition 5.2. *Let G be an undirected, connected graph and Z be a subset of at most 4 terminals. If no two central cuts have the same capacity, then the unique GH Z-Tree T occurs as a weak bag minor. Moreover, if T is a path, then it occurs as a bag minor.*

Proof. We first consider the case where we have 4 terminals and let T be the unique GH tree. Suppose that T is a star with center node 1 and let B_1, B_2, B_3, B_4 be the bags. Since each fundamental cut of T is central (in G) we have that B_2, B_3, B_4 each induces a connected subgraph of G. Let $Y \subseteq B_1$ be those nodes (if any) which do not lie in the same component of $G[B_1]$ as 1. We may try to produce T as a weak bag minor of G by deleting Y. This fails only if for some $j \geq 2$, $c(B_1 \setminus Y, B_j) = 0$ (the only real edges between B_j, B_1 are incident to Y). Suppose this occurs for say $j = 2$ (the other cases are the same). Let $R = B_2 \cup Y \cup B_3, S = B_2 \cup Y \cup B_4$. It follows that $c(R \cap S, V - (R \cup S)) = 0$ and hence $c(R) + c(S) = c(R \setminus S) + c(S \setminus R) = c(B_3) + c(B_4)$. But $\delta(R)$ is a 34-cut distinct from $\delta(B_3)$. Hence by uniqueness of minimum cuts, $c(R) > c(B_3)$. Similarly, $c(S) > c(B_4)$. This is contradiction, thus completing the first part.

Consider now the case where T is a path, say $1, 2, 3, 4$. Since each fundamental cut is central, $G[B_1], G[B_4]$ are connected. Now suppose that $G[B_2]$ is not connected. Let M be the set of nodes which do not lie in the same component as 2. If we define $X = B_1, Y = B_3 \cup B_4$ and $t = 2, x = 1, y = 3$, then Lemma 1.12 implies that $c(M, B_2 \setminus M) > 0$ a contradiction. It remains to show that $c(B_i, B_{i+1}) > 0$ for each $i = 1, 2, 3$.

Suppose first that $c(B_1, B_2) = 0$. Then $c(B_1 \cup B_3 \cup B_4) \leq c(B_3 \cup B_4)$ contradicting the fact that $B_3 \cup B_4$ induces the unique minimum 23 cut. Hence $c(B_1, B_2) > 0$ and by symmetry $c(B_3, B_4) > 0$. Finally suppose that $c(B_2, B_3) = 0$. Then, one easily checks that $c(B_1) + c(B_4) \geq c(B_2) + c(B_3)$. But then either B_2 induces a second minimum 12 cut, or B_3 induces another minimum 34 cut. In either case, we have a contradiction. The final cases where $|Z| \leq 3$ follow easily by the same methods. ∎

Lemma 5.3. *Let T be a GH Z-Tree bag minor for some capacitated graph G and let $v \in Z$. Let $uv \in T$ be the maximum weight edge of T incident to v. If we set $B'(u) = B(v) \cup B(u)$ and $B'(x) = B(x)$ for each $x \in Z \setminus \{u, v\}$, then the resulting partition defines a GH $(Z \setminus v)$-Tree T' which is a bag minor.*

Proof. Clearly T' is a bag minor and every fundamental cut of T, other than uv's, is still a fundamental cut of T' – see Figure A. It remains to show that for any $a, b \in Z \setminus v$, there is a minimum

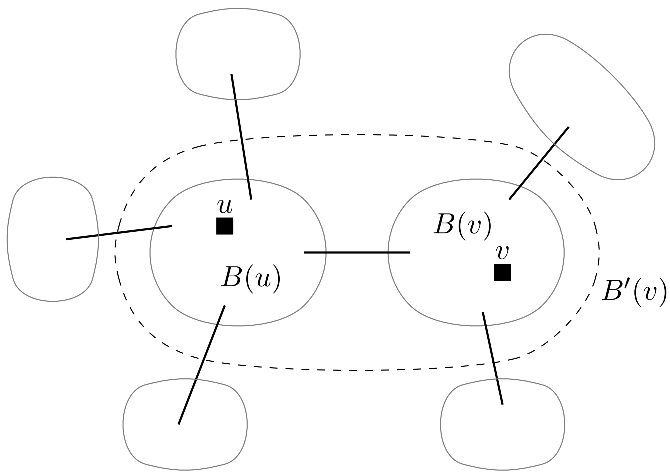

Figure 10: Illustration for Lemma 5.3.

ab-cut that does not correspond to the fundamental cut of uv. This is immediate if the unique ab-path P in T does not contain uv. If it does contain uv, then since $a, b \neq v$, the ab-path in T contains some edge vw. But since $c'(vw) \leq c'(uv)$, the result follows. ■